NEW OUTLOOK ON THE MINIMAL MODEL PROGRAM, I

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Abstract. We give a new and self-contained proof of the finite generation of adjoint rings with big boundaries. As a consequence, we show that the canonical ring of a smooth projective variety is finitely generated.

Contents

1. Introduction

The main goal of this paper is to provide a new proof of the following theorem while avoiding techniques of the Minimal Model Program.

Theorem 1.1. *Let X be a smooth projective variety and let Δ be a $\mathbb{Q}$-divisor with simple normal crossings such that $\lfloor \Delta \rfloor = 0$.*

Then the log canonical ring $R(X, K_X + \Delta)$ is finitely generated.

Part of this work was written while the second author was a PhD student of A. Corti, who influenced ideas developed here immensely. Part of the paper started as a collaboration with J. M^{c}Kernan. We would like to express our gratitude to both of them for their encouragement, support and continuous inspiration. We thank F. Ambro, C. Hacon, J. Hausen, A.-S. Kaloghiros, A. Lopez, K. Matsuki, and M. Reid for many useful comments. We are particularly grateful to the referees who helped to improve the presentation of the paper considerably.

The first author was partially supported by an EPSRC grant. The second author is grateful for support from the University of Cambridge, the Max-Planck-Institut für Mathematik, and the Institut Fourier.

This work supersedes [Laz09], where the results of this paper were first proved without the Minimal Model Program by the second author. Several arguments here follow closely those in [Laz09] and, based on these methods, we obtain a streamlined proof which is almost entirely self-contained. We even prove a lifting statement for adjoint bundles without relying on asymptotic multiplier ideals, assuming only Kawamata-Viehweg vanishing and some elementary arithmetic – this is Theorem 3.4, which slightly generalises the lifting theorem from [HM10].

The results presented here were originally proved by extensive use of methods of the Minimal Model Program in [BCHM10, HM10], and an analytic proof of finite generation of the canonical ring for varieties of general type is announced in [Siu08]. By contrast, in this paper we avoid the following tools which are commonly used in the Minimal Model Program: Mori's bend and break, which relies on methods in positive characteristic [Mor82], the Cone and Contraction theorem [KM98], the theory of asymptotic multiplier ideals, which was necessary to prove the existence of flips in [HM10]. Moreover, contrary to classical Minimal Model Program, we do not need to work with singular varieties.

In [CL10], Corti and the second author recently proved that the Cone and Contraction theorem, and the main result of [BCHM10], follow quickly from one of our main results, Theorem A. Therefore, this paper and [CL10] together give a completely new organisation of the Minimal Model Program.

We now briefly describe the strategy of the proof. As part of the induction, we prove the following two theorems.

Theorem A. *Let X be a smooth projective variety of dimension n. Let $B_1, \dots, B_k$ be $\mathbb{Q}$-divisors on X such that $\lfloor B_i \rfloor = 0$ for all i, and such that the support of $\sum_{i=1}^k B_i$ has simple normal crossings. Let A be an ample $\mathbb{Q}$-divisor on X, and denote $D_i = K_X + A + B_i$ for every i.*

Then the adjoint ring

$$R(X; D_1, \dots, D_k) = \bigoplus_{(m_1, \dots, m_k) \in \mathbb{N}^k} H^0\big(X, \mathcal{O}_X\big(\lfloor \sum m_i D_i \rfloor\big)\big)$$

is finitely generated.

Theorem B. *Let $(X, \sum_{i=1}^p S_i)$ be a log smooth projective pair of dimension n, where $S_1, \dots, S_p$ are distinct prime divisors. Let $V = \sum_{i=1}^p \mathbb{R} S_i \subseteq \mathrm{Div}_{\mathbb{R}}(X)$, let $\mathcal{L}(V) = \{B = \sum b_i S_i \in V \mid 0 \leq b_i \leq 1 \text{ for all } i\}$, and let A be an ample $\mathbb{Q}$-divisor on X.*

Then

$$\mathcal{E}_A(V) = \{B \in \mathcal{L}(V) \mid |K_X + A + B|_{\mathbb{R}} \neq \emptyset\}$$

is a rational polytope.

Note that all the results in this paper hold, with the same proofs, when varieties are projective over affine varieties. For definitions of various terms involved in the statements of the theorems, see Section 2. In the sequel, "Theorem A_n" stands for "Theorem A in dimension n," and so forth.

In Section 2 we lay the foundation for the remainder of the paper: we discuss basic properties of asymptotic invariants of divisors, convex geometry and Diophantine approximation, and we introduce divisorial rings graded by monoids of higher rank and present basic consequences of finite generation of these rings. Basic references for asymptotic invariants of divisors are [Nak04, ELM$^+$06]. The first systematic use of Diophantine approximation in the Minimal Model Program was initiated by Shokurov in [Sho03], and our arguments at several places in this paper are inspired by some of the techniques introduced there.

In Section 3 we give a simplified proof of a version of the lifting lemma from [HM10]. The proof in [HM10] is based on methods initiated in [Siu98], which also inspired a systematic use of multiplier ideals. We want to emphasise that our proof, even though ultimately following the same path, is much simpler and uses only Kawamata-Viehweg vanishing and some elementary arithmetic.

In Section 4 we prove that one of the sets which naturally appears in the theory is a rational polytope. Some steps in the proof are close in spirit to Hacon's ideas in the proof of [HK10, Theorem 9.16]. The proof is an application of the lifting result from Section 3.

In Section 5 we prove Theorem B_n, assuming Theorems A_{n-1} and B_{n-1}. Certain steps of the proof here are similar to [BCHM10, Section 6], and they rely on Nakayama's techniques from [Nak04]. Lemma 5.3 was obtained in [Pău08] by analytic methods, without assuming Theorems A_{n-1} and B_{n-1}. We remark here that several arguments of this section can be made somewhat shorter if one were to assume some facts about lengths of extremal rays, similarly as in [BCHM10]; however, we are deliberately making the proofs a bit longer by proving everything "from scratch", especially since one of the aims of this paper is to provide the basis for simpler proofs of the foundational results of the Minimal Model Program [CL10].

Finally, in Section 6, we prove Theorem A_n, assuming Theorems A_{n-1} and B_n, therefore completing the induction step. This part of the proof is close in spirit to that of the finite generation of the restricted ring when the grading is by the non-negative integers, see [Cor07, Lemma 2.3.6].

The papers [Cor11] and [CL11] give an introduction to some of the ideas presented in this work.

2. Preliminary results

2.1. **Notation and conventions.** In this paper all algebraic varieties are defined over $\mathbb{C}$. We denote by $\mathbb{R}_+$ and $\mathbb{Q}_+$ the sets of non-negative real and rational numbers. For any $x, y \in \mathbb{R}^N$, we denote by $[x, y]$ and (x, y) the closed and open segments joining x and y. Given subsets $A, B \subseteq \mathbb{R}^N$, the Minkowski sum of A and B is

$$A + B = \{a + b \mid a \in A,\, b \in B\}.$$

We denote by $\overline{\mathcal{C}}$ the topological closure of a set $\mathcal{C} \subseteq \mathbb{R}^N$.

Let X be a smooth projective variety and $\mathbf{R} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$. We denote by $\mathrm{Div}_{\mathbf{R}}(X)$ the group of $\mathbf{R}$-divisors on X, and $\sim_{\mathbf{R}}$ and $\equiv$ denote $\mathbf{R}$-linear and numerical equivalence of $\mathbb{R}$-divisors. If $A = \sum a_i C_i$ and $B = \sum b_i C_i$ are two $\mathbb{R}$-divisors on X, then $\lfloor A \rfloor = \sum \lfloor a_i \rfloor C_i$ is the round-down of A, $\lceil A \rceil = \sum \lceil a_i \rceil C_i$ is the round-up of A, $\{A\} = A - \lfloor A \rfloor$ is the fractional part of A, $\|A\| = \max\limits_i \{|a_i|\}$ is the sup-norm of A, and

$$A \wedge B = \sum \min\{a_i, b_i\} C_i.$$

Given $D \in \mathrm{Div}_{\mathbb{R}}(X)$ and $x \in X$, $\mathrm{mult}_x D$ is the order of vanishing of D at x. If S is a prime divisor, $\mathrm{mult}_S D$ is the order of vanishing of D at the generic point of S.

In this paper, a *log pair* (X, Δ) consists of a smooth variety X and an $\mathbb{R}$-divisor $\Delta \geq 0$. We say that (X, Δ) is *log smooth* if $\mathrm{Supp}\, \Delta$ has simple normal crossings. A projective birational morphism $f \colon Y \longrightarrow X$ is a *log resolution* of the pair (X, Δ) if Y is smooth, $\mathrm{Exc}\, f$ is a divisor and the support of $f_*^{-1}\Delta + \mathrm{Exc}\, f$ has simple normal crossings.

Definition 2.1. Let (X, Δ) be a log pair with $\lfloor \Delta \rfloor = 0$. Then (X, Δ) has *klt* (respectively *canonical*, *terminal*) singularities if for every log resolution $f \colon Y \longrightarrow X$, if we write $E = K_Y + f_*^{-1}\Delta - f^*(K_X + \Delta)$, we have $\lceil E \rceil \geq 0$ (respectively $E \geq 0$; $E \geq 0$ and $\mathrm{Supp}\, E = \mathrm{Exc}\, f$). Note that if (X, Δ) is terminal, then for every $\mathbb{R}$-divisor G, the pair $(X, \Delta + \varepsilon G)$ is also terminal for every $0 \leq \varepsilon \ll 1$.

The following result is standard.

Lemma 2.2. *Let $(X, S + B)$ be a log smooth projective pair, where S is a prime divisor and B is a $\mathbb{Q}$-divisor such that $\lfloor B \rfloor = 0$ and $S \not\subseteq \mathrm{Supp}\, B$. Then there exist a log resolution $f \colon Y \longrightarrow X$ of $(X, S + B)$ and $\mathbb{Q}$-divisors $C, E \geq 0$ on Y with no common components, such that the components of C are disjoint, E is f-exceptional, and if $T = f_*^{-1} S$, then*

$$K_Y + T + C = f^*(K_X + S + B) + E.$$

Proof. By [KM98, Proposition 2.36], there exist a log resolution $f \colon Y \longrightarrow X$ which is a sequence of blow-ups along intersections of components of B, and $\mathbb{Q}$-divisors $C, E \geq 0$ on Y with no common components, such that the components of C are disjoint, E is f-exceptional, and

$$K_Y + C = f^*(K_X + B) + E.$$

Since $(X, S + B)$ is log smooth, it follows that if some components of B intersect, then no irreducible component of their intersection is contained in S. Thus $T = f^* S$, and the lemma follows. $\square$

If X is a smooth projective variety, and if D is an integral divisor on X, we denote by $\mathrm{Bs}\,|D|$ the base locus of D. If D is an $\mathbb{R}$-divisor on X, we denote

$$|D|_{\mathbf{R}} = \{D' \geq 0 \mid D \sim_{\mathbf{R}} D'\} \qquad \text{and} \qquad \mathbf{B}(D) = \bigcap_{D' \in |D|_{\mathbb{R}}} \mathrm{Supp}\, D',$$

and we call $\mathbf{B}(D)$ the *stable base locus* of D. We set $\mathbf{B}(D) = X$ if $|D|_{\mathbb{R}} = \emptyset$. The following result shows that this is compatible with the usual definition, see [BCHM10, Lemma 3.5.3].

Lemma 2.3. *Let X be a smooth projective variety and let D be a $\mathbb{Q}$-divisor. Then $\mathbf{B}(D) = \bigcap_q \mathrm{Bs}\,|qD|$ for all q sufficiently divisible.*

Proof. Fix a point $x \in X \setminus \mathbf{B}(D)$. Then there exist an $\mathbb{R}$-divisor $F \geq 0$, real numbers $r_1, \dots, r_k$ and rational functions $f_1, \dots, f_k \in k(X)$ such that $F = D + \sum_{i=1}^k r_i(f_i)$ and $x \notin \mathrm{Supp}\, F$. Let $W \subseteq \mathrm{Div}_{\mathbb{R}}(X)$ be the subspace spanned by the components of D and all (f_i). Let $W_0 \subseteq W$ be the subspace of divisors $\mathbb{R}$-linearly equivalent to zero, and note that W_0 is a rational subspace of W. Consider the quotient map $\pi\colon W \longrightarrow W/W_0$. Then the set $\{G \in \pi^{-1}(\pi(D)) \mid G \geq 0\}$ is not empty as it contains F, and it is cut out from W by rational hyperplanes. Thus, it contains a $\mathbb{Q}$-divisor $D' \geq 0$ such that $D \sim_{\mathbb{Q}} D'$ and $x \notin \mathrm{Supp}\, D'$. $\square$

Definition 2.4. Let $(X, S + \sum_{i=1}^p S_i)$ be a log smooth projective pair, where S and all S_i are distinct prime divisors, let $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \mathrm{Div}_{\mathbb{R}}(X)$, and let A be a $\mathbb{Q}$-divisor on X. We define

$$\begin{aligned}
\mathcal{L}(V) &= \{B = \textstyle\sum b_i S_i \in V \mid 0 \leq b_i \leq 1 \text{ for all } i\},\\
\mathcal{E}_A(V) &= \{B \in \mathcal{L}(V) \mid |K_X + A + B|_{\mathbb{R}} \neq \emptyset\},\\
\mathcal{B}_A^S(V) &= \{B \in \mathcal{L}(V) \mid S \not\subseteq \mathbf{B}(K_X + S + A + B)\}.
\end{aligned}$$

If D is an integral divisor, $\mathrm{Fix}\,|D|$ and $\mathrm{Mob}(D)$ denote the *fixed* and *mobile* parts of D. Hence $|D| = |\mathrm{Mob}(D)| + \mathrm{Fix}\,|D|$, and the base locus of $|\mathrm{Mob}(D)|$ contains no divisors. More generally, if V is any linear system on X, $\mathrm{Fix}(V)$ denotes the fixed divisor of V. If S is a prime divisor on X such that $S \not\subseteq \mathrm{Bs}\,|D|$, then $|D|_S$ denotes the image of the linear system $|D|$ under restriction to S.

Definition 2.5. Let X be a smooth projective variety and let S be a smooth prime divisor. Let C and D be $\mathbb{Q}$-divisors on X such that $|C|_{\mathbb{Q}} \neq \emptyset$, $|D|_{\mathbb{Q}} \neq \emptyset$ and $S \not\subseteq \mathbf{B}(D)$. Then by Lemma 2.3, we may define

$$\mathbf{Fix}(C) = \liminf \frac{1}{k}\,\mathrm{Fix}\,|kC| \qquad \text{and} \qquad \mathbf{Fix}_S(D) = \liminf \frac{1}{k}\,\mathrm{Fix}\,|kD|_S$$

for all k sufficiently divisible.

2.2. **Convex geometry and Diophantine approximation.**

Definition 2.6. Let $\mathcal{C} \subseteq \mathbb{R}^N$ be a convex set. A subset $F \subseteq \mathcal{C}$ is a *face* of $\mathcal{C}$ if F is convex, and whenever $tu + (1-t)v \in F$ for some $u, v \in \mathcal{C}$ and $0 < t < 1$, then $u, v \in F$. Note that $\mathcal{C}$ is itself a face of $\mathcal{C}$. We say that $x \in \mathcal{C}$ is an *extreme point* of $\mathcal{C}$ if $\{x\}$ is a face of $\mathcal{C}$. For $y \in \mathcal{C}$, the minimal face of $\mathcal{C}$ which contains y is denoted by $\operatorname{face}(\mathcal{C}, y)$. It is a well known fact that any compact convex set $\mathcal{C} \subseteq \mathbb{R}^N$ is the convex hull of its extreme points.

A *polytope* in $\mathbb{R}^N$ is a compact set which is the intersection of finitely many half spaces; equivalently, it is the convex hull of finitely many points in $\mathbb{R}^N$. A polytope is *rational* if it is an intersection of finitely many rational half spaces; equivalently, it is the convex hull of finitely many rational points in $\mathbb{R}^N$. A *rational polyhedral cone* in $\mathbb{R}^N$ is a convex cone spanned by finitely many rational vectors.

Remark 2.7. Given a smooth projective variety X, we often consider subspaces $V \subseteq \operatorname{Div}_{\mathbb{R}}(X)$ which are spanned by a finite set of prime divisors. Thus, these divisors implicitly define an isomorphism between V and $\mathbb{R}^N$ for some N.

With notation from Definition 2.4, $\mathcal{L}(V)$ is a rational polytope. Also, the set of rational points is dense in $\mathcal{B}_A^S(V)$. Indeed, if $B = \sum b_i S_i \in \mathcal{B}_A^S(V)$, then $B + \sum\limits_{b_i<1} \varepsilon_i S_i \in \mathcal{B}_A^S(V)$ for all $0 \leq \varepsilon_i \ll 1$.

Lemma 2.8. *Let $\mathcal{P}$ be a compact convex set in $\mathbb{R}^N$, and fix any norm $\|\cdot\|$ on $\mathbb{R}^N$.*

Then $\mathcal{P}$ is a polytope if and only if for every point $x \in \mathcal{P}$ there exists a real number $\delta = \delta(x, \mathcal{P}) > 0$, such that for every $y \in \mathbb{R}^N$ with $0 < \|x - y\| < \delta$, if $(x, y) \cap \mathcal{P} \neq \emptyset$, then $y \in \mathcal{P}$.

Proof. Suppose that $\mathcal{P}$ is a polytope and let $x \in \mathcal{P}$. Let $F_1, \ldots, F_k$ be the set of all the faces of $\mathcal{P}$ which do not contain x. Then it is enough to define

$$\delta(x, \mathcal{P}) = \min\{\|x - y\| \mid y \in F_i \text{ for some } i = 1, \ldots, k\}.$$

Conversely, assume that $\mathcal{P}$ is not a polytope, and let x_n be an infinite sequence of distinct extreme points of $\mathcal{P}$. Since $\mathcal{P}$ is compact, by passing to a subsequence we may assume that there exists $x = \lim\limits_{n\to\infty} x_n \in \mathcal{P}$. For any real number $\delta > 0$ pick $k \in \mathbb{N}$ such that $0 < \|x - x_k\| < \delta$, and set $x' = x + t(x_k - x)$ for some $1 < t < \delta/\|x - x_k\|$. Then $0 < \|x - x'\| < \delta$ and $\emptyset \neq (x, x_k) \subseteq (x, x') \cap \mathcal{P}$, but $x' \notin \mathcal{P}$ since x_k is an extreme point of $\mathcal{P}$. This proves the lemma. $\square$

Remark 2.9. With assumptions from Lemma 2.8, assume additionally that $\mathcal{P}$ does not contain the origin, and let $\mathcal{C} = \mathbb{R}_+\mathcal{P}$. Then the same proof shows that $\mathcal{C}$ is a polyhedral cone if and only if for every point $x \in \mathcal{C}$ there exists a real number $\delta = \delta(x, \mathcal{C}) > 0$, such that for every $y \in \mathbb{R}^N$ with $0 < \|x - y\| < \delta$, if $(x, y) \cap \mathcal{C} \neq \emptyset$, then $y \in \mathcal{C}$.

Lemma 2.10. *Let $\mathcal{P} \subseteq \mathbb{R}^N$ be a polytope which does not contain the origin, and let $\mathcal{D} = \mathbb{R}_+\mathcal{P}$. Let $\Sigma \in \mathcal{D}\backslash\{0\}$ and let $\Sigma_m \in \mathbb{R}^N$ be a sequence of distinct points such that $\lim\limits_{m\to\infty} \Sigma_m = \Sigma$. Let $S \in \mathbb{R}^N\backslash\{0\}$, let $c_m \geq 0$ be a bounded sequence of real numbers, and set $\Gamma_m = \Sigma_m - c_m S$. Assume that $(\Sigma, \Gamma_m) \cap \mathcal{D} \neq \emptyset$ for every $m \in \mathbb{N}$.*

Then, there exists $P_m \in [\Sigma_m, \Gamma_m] \cap \mathcal{D}$ for infinitely many m. If additionally $P_m = \Gamma_m$ and $(\Sigma_m, \Gamma_m) \cap \mathcal{D} = \emptyset$ for all m, then after passing to a subsequence, we have $\lim\limits_{m\to\infty} \Gamma_m = \Sigma$.

Proof. Fix any norm $\|\cdot\|$ on $\mathbb{R}^N$. By passing to a subsequence we may assume that there is a constant $c \geq 0$ such that $c = \lim\limits_{m\to\infty} c_m$. Assume first that $c = 0$. Then $\lim\limits_{m\to\infty} \Gamma_m = \Sigma$, and since $\mathcal{D}$ is a polyhedral cone, Remark 2.9 implies that $\Gamma_m \in \mathcal{D}$ for $m \gg 0$ and the lemma follows.

Thus, from now on we assume that $c > 0$. First we consider the case when there are infinitely many m such that $\Sigma_m \in \Sigma + \mathbb{R}S$. Then there is a fixed point $P_m = P \in (\Sigma, \Gamma_m) \cap \mathcal{D}$ for all $m \gg 0$. Since $\lim\limits_{m\to\infty} \Sigma_m = \Sigma$, it follows that $P \in [\Sigma_m, \Gamma_m] \cap \mathcal{D}$ for all m and the first claim follows. For the second claim, if additionally $(\Sigma_m, \Gamma_m) \cap \mathcal{D} = \emptyset$, then $P = \Sigma_m$ since $P \neq \Gamma_m$ by definition of P, and we immediately get a contradiction.

Therefore, we may assume that $\Sigma_m \notin \Sigma + \mathbb{R}S$ for all m. By Remark 2.9 there exist points $Q_m \in (\Sigma, \Gamma_m) \cap \mathcal{D}$ and a constant $0 < d < c$ such that $\|Q_m - \Sigma\| = d$ for all $m \gg 0$. After passing to a subsequence, we may assume that there exists $\lim\limits_{m\to\infty} Q_m = Q \in \mathcal{D}$. Note that $Q \in [\Sigma, \Sigma - cS]$. For every m, as

$$\Gamma_m = \frac{c_m}{c}\Sigma - \frac{c_m}{c}(\Sigma - cS) + \Sigma_m,$$

Γ_m belongs to the affine 2-plane $\{t_1\Sigma + t_2(\Sigma - cS) + t_3\Sigma_m \mid t_i \in \mathbb{R}, t_1 + t_2 + t_3 = 1\}$, and since $d < c$, for all $m \gg 0$ there exist $P_m \in [\Sigma_m, \Gamma_m]$ such that $Q_m \in [P_m, Q]$. It is easy to see that $\lim\limits_{m\to\infty} P_m = Q$ and by Remark 2.9 it follows that $P_m \in \mathcal{D}$ for $m \gg 0$, as claimed.

Now assume additionally that $P_m = \Gamma_m$ and $(\Sigma_m, \Gamma_m) \cap \mathcal{D} = \emptyset$, and observe that $\lim\limits_{m\to\infty} \Gamma_m = \Sigma - cS \neq \Sigma$. Denote $\Gamma = \Sigma - \frac{c}{2}S \in \mathcal{D}$ and $R_m = \Gamma_m + \frac{c}{2}S$; note that $R_m \in (\Sigma_m, \Gamma_m)$ for $m \gg 0$, and thus $R_m \notin \mathcal{D}$. Let $\delta = \delta(\Gamma, \mathcal{D}) > 0$, whose existence is guaranteed by Remark 2.9, and pick $m \gg 0$ such that $\|R_m - \Gamma\| = \|\Gamma_m - (\Sigma - cS)\| < \delta$ and $R_m \notin \mathcal{D}$. Then the segments $[\Sigma, \Gamma_m]$ and $[R_m, \Gamma]$ intersect at a point $R'_m \neq \Gamma$, and we have $R'_m \in \mathcal{D}$ since $[\Gamma_m, \Sigma] = [P_m, \Sigma] \subseteq \mathcal{D}$. But then $R_m \in \mathcal{D}$ by Remark 2.9, a contradiction. Thus $c = 0$ and $\lim\limits_{m\to\infty} \Gamma_m = \Sigma$. □

Lemma 2.11 (Gordan's Lemma)**.** *Let $\mathcal{C} \subseteq \mathbb{R}^N$ be a rational polyhedral cone. Then $\mathcal{C} \cap \mathbb{Z}^N$ is a finitely generated monoid.*

Proof. See [Ful93, §1.2]. □

Definition 2.12. Let $\mathcal{C} \subseteq \mathbb{R}^N$ be a convex set and let $\Phi\colon \mathcal{C} \longrightarrow \mathbb{R}$ be a function. Then Φ is *convex* if $\Phi\big(tx + (1-t)y\big) \leq t\Phi(x) + (1-t)\Phi(y)$ for any $x, y \in \mathcal{C}$ and any $t \in [0,1]$. If $\mathcal{C}$ is a rational polytope, then Φ is *rationally piecewise affine* if there exists a finite decomposition $\mathcal{C} = \bigcup_{i=1}^{\ell} \mathcal{C}_i$ into rational polytopes such that $\Phi_{|\mathcal{C}_i}$ is a rational affine map for all i. If $\mathcal{C}$ is a cone, then Φ is *homogeneous of degree one* if $\Phi(tx) = t\Phi(x)$ for any $x \in \mathcal{C}$ and $t \in \mathbb{R}_+$.

Lemma 2.13. *Let $\mathcal{H} \subseteq \mathbb{R}^N$ be a rational affine hyperplane which does not contain the origin, and let $\mathcal{P} \subseteq \mathcal{H}$ be a rational polytope. Let $\mathcal{P}_{\mathbb{Q}} = \mathcal{P} \cap \mathbb{Q}^N$, and let $f\colon \mathcal{P}_{\mathbb{Q}} \longrightarrow \mathbb{R}$ be a bounded convex function. Assume that there exist $x_1, \dots, x_q \in \mathcal{P}_{\mathbb{Q}}$ with $f(x_i) \in \mathbb{Q}$ for all i, and that for any $x \in \mathcal{P}_{\mathbb{Q}}$ there exists $(r_1, \dots, r_q) \in \mathbb{R}_+^q$ such that $x = \sum r_i x_i$ and $f(x) = \sum r_i f(x_i)$.*

Then f can be extended to a rational piecewise affine function on $\mathcal{P}$.

Proof. Since $\mathcal{P} \subseteq \mathcal{H}$, for any $x \in \mathcal{P}_{\mathbb{Q}}$ and $(r_1, \dots, r_q) \in \mathbb{R}_+^q$ such that $x = \sum r_i x_i$, we have $\sum r_i = 1$. Pick $C \in \mathbb{Q}_+$ such that $-C \leq f(x) \leq C$ for all $x \in \mathcal{P}_{\mathbb{Q}}$.

Let $\mathcal{Q} \subseteq \mathbb{R}^{N+1}$ be the convex hull of all the points $\big(x_i, f(x_i)\big)$ and (x_i, C), and set $\mathcal{Q}' = \{(x,y) \in \mathcal{P}_{\mathbb{Q}} \times \mathbb{R} \mid f(x) \leq y \leq C\}$. Since f is convex, and all $\big(x_i, f(x_i)\big)$ and (x_i, C) are contained in $\mathcal{Q}'$, it follows that $\mathcal{Q} \cap \mathbb{Q}^{N+1} \subseteq \mathcal{Q}'$. Now, fix $(u,v) \in \mathcal{Q}'$. Then there exists $t \in [0,1]$ such that $v = tf(u) + (1-t)C$, and as $u \in \mathcal{P}_{\mathbb{Q}}$, there exist $r_i \in \mathbb{R}_+$ such that $\sum r_i = 1$, $u = \sum r_i x_i$ and $f(u) = \sum r_i f(x_i)$. Therefore

$$(u,v) = \sum t r_i \big(x_i, f(x_i)\big) + \sum (1-t) r_i (x_i, C),$$

and hence $(u,v) \in \mathcal{Q}$. This yields $\mathcal{Q} \cap \mathbb{Q}^{N+1} = \mathcal{Q}' \cap \mathbb{Q}^{N+1}$, and in particular $\mathcal{Q} = \overline{\mathcal{Q}'}$.

Define $F\colon \mathcal{P} \longrightarrow [-C, C]$ as

$$F(x) = \min\{y \in [-C, C] \mid (x,y) \in \mathcal{Q}\}.$$

Then F extends f, and it is rational piecewise affine as $\mathcal{Q}$ is a rational polytope. $\square$

We use the following result from Diophantine approximation.

Lemma 2.14. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^N$, let $\mathcal{P} \subseteq \mathbb{R}^N$ be a rational polytope and let $x \in \mathcal{P}$. Fix a positive integer k and a positive real number ε.*

Then there are finitely many $x_i \in \mathcal{P}$ and positive integers k_i divisible by k, such that $k_i x_i / k$ are integral, $\|x - x_i\| < \varepsilon / k_i$, and x is a convex linear combination of x_i.

Proof. See [BCHM10, Lemma 3.7.7]. $\square$

2.3. **Nakayama-Zariski decomposition.** We need several definitions and results from [Nak04].

Definition 2.15. Let X be a smooth projective variety, let A be an ample $\mathbb{R}$-divisor, and let Γ be a prime divisor. If $D \in \mathrm{Div}_{\mathbb{R}}(X)$ is a big divisor, define

$$o_\Gamma(D) = \inf\{\operatorname{mult}_\Gamma D' \mid D' \in |D|_{\mathbb{R}}\}.$$

If $D \in \operatorname{Div}_{\mathbb{R}}(X)$ is pseudo-effective, set

$$\sigma_\Gamma(D) = \lim_{\varepsilon \to 0} o_\Gamma(D + \varepsilon A) \qquad \text{and} \qquad N_\sigma(D) = \textstyle\sum_\Gamma \sigma_\Gamma(D) \cdot \Gamma,$$

where the sum runs over all prime divisors Γ on X.

Lemma 2.16. *Let X be a smooth projective variety, let A be an ample $\mathbb{R}$-divisor, let D be a pseudo-effective $\mathbb{R}$-divisor, and let Γ be a prime divisor. Then $\sigma_\Gamma(D)$ exists as a limit, it is independent of the choice of A, it depends only on the numerical equivalence class of D, and $\sigma_\Gamma(D) = o_\Gamma(D)$ if D is big. The function σ_Γ is homogeneous of degree one, convex and lower semi-continuous on the cone of pseudo-effective divisors on X, and it is continuous on the cone of big divisors. For every pseudo-effective $\mathbb{R}$-divisor E we have $\sigma_\Gamma(D) = \lim\limits_{\varepsilon \to 0} \sigma_\Gamma(D + \varepsilon E)$.*

Furthermore, $N_\sigma(D)$ is an $\mathbb{R}$-divisor on X, $D - N_\sigma(D)$ is pseudo-effective, and for any $\mathbb{R}$-divisor $0 \leq F \leq N_\sigma(D)$ we have $N_\sigma(D - F) = N_\sigma(D) - F$.

Proof. See [Nak04, §III.1]. $\square$

Remark 2.17. Let X be a smooth projective variety, let D_m be a sequence of pseudo-effective $\mathbb{R}$-divisors which converge to an $\mathbb{R}$-divisor D, and let Γ be a prime divisor on X. Then the sequence $\sigma_\Gamma(D_m)$ is bounded. Indeed, pick $k \gg 0$ such that $D - k\Gamma$ is not pseudo-effective, and assume that $\sigma_\Gamma(D_m) > k$ for infinitely many m. Then $D_m - k\Gamma$ is pseudo-effective for infinitely many m by Lemma 2.16, a contradiction.

Remark 2.18. Let X be a smooth projective variety, let D be a pseudo-effective $\mathbb{R}$-divisor, let A be an ample $\mathbb{R}$-divisor, and let $x \in X \setminus \bigcup_{\varepsilon > 0} \mathbf{B}(D + \varepsilon A)$. Let $f \colon Y \to X$ be the blowup of X along x with the exceptional divisor E. Then $\sigma_E(f^*D) = 0$. To see this, observe that $E \not\subseteq \mathbf{B}(f^*D + \varepsilon f^*A)$, and thus $o_E(f^*D + \varepsilon f^*A) = 0$. Letting $\varepsilon \to 0$, we conclude by Lemma 2.16.

Lemma 2.19. *Let X be a smooth projective variety, let D be a pseudo-effective $\mathbb{R}$-divisor, and let A be an ample $\mathbb{Q}$-divisor.*

If $D \not\equiv N_\sigma(D)$, then there exist a positive integer k and a positive rational number β such that kA is integral and

$$h^0(X, \mathcal{O}_X(\lfloor mD \rfloor + kA)) > \beta m \quad \text{for all} \quad m \gg 0.$$

Proof. Replacing D by $D - N_\sigma(D)$, we may assume that $N_\sigma(D) = 0$. Now apply [Nak04, Theorem V.1.11]. $\square$

Lemma 2.20. *Let X be a smooth projective variety, let D be a pseudo-effective $\mathbb{R}$-divisor on X, and let $\Gamma_1, \ldots, \Gamma_\ell$ be distinct prime divisors such that $\sigma_{\Gamma_i}(D) > 0$ for all i.*

Then for any $\gamma_j \in \mathbb{R}_+$ we have $\sigma_{\Gamma_i}(\sum_{j=1}^{\ell} \gamma_j \Gamma_j) = \gamma_i$ for every i. In particular, if $D \geq 0$ and if $\sigma_\Gamma(D) > 0$ for every component Γ of D, then $D = N_\sigma(D)$.

Proof. This is [Nak04, Proposition III.1.10]. □

Lemma 2.21. *Let X be a smooth projective variety and let Γ be a prime divisor. Let D be a pseudo-effective $\mathbb{R}$-divisor and let A be an ample $\mathbb{R}$-divisor.*

(i) If $\sigma_\Gamma(D) = 0$, then $\Gamma \nsubseteq \mathbf{B}(D + A)$.
(ii) If $\sigma_\Gamma(D) > 0$, then $\Gamma \subseteq \mathbf{B}(D + \varepsilon A)$ for $0 < \varepsilon \ll 1$.

Proof. For (i), note that $\sigma_\Gamma(D + \frac{1}{2}A) \leq \sigma_\Gamma(D) = 0$. By Lemma 2.16 there exists $0 \leq D' \sim_\mathbb{R} D + \frac{1}{2}A$ such that $\gamma = \operatorname{mult}_\Gamma D' \ll 1$, and in particular $\frac{1}{2}A + \gamma\Gamma$ is ample. Pick $A' \sim_\mathbb{R} \frac{1}{2}A + \gamma\Gamma$ such that $A' \geq 0$ and $\operatorname{mult}_\Gamma A' = 0$. Then

$$D + A \sim_\mathbb{R} D' - \gamma\Gamma + A' \geq 0 \qquad \text{and} \qquad \operatorname{mult}_\Gamma(D' - \gamma\Gamma + A') = 0.$$

This proves the first claim. The second claim follows from $0 < \sigma_\Gamma(D) = \lim\limits_{\varepsilon \to 0} o_\Gamma(D + \varepsilon A)$, since then $o_\Gamma(D + \varepsilon A) > 0$ for $0 < \varepsilon \ll 1$. □

2.4. **Divisorial rings.** Now we establish properties of finite generation of (divisorial) graded rings that we use in the paper.

Definition 2.22. Let X be a smooth projective variety and let $\mathcal{S} \subseteq \operatorname{Div}_\mathbb{Q}(X)$ be a finitely generated monoid. Then

$$R(X, \mathcal{S}) = \bigoplus_{D \in \mathcal{S}} H^0\big(X, \mathcal{O}_X(\lfloor D \rfloor)\big)$$

is a *divisorial $\mathcal{S}$-graded ring*. If $D_1, \dots, D_\ell$ are generators of $\mathcal{S}$ and if $D_i \sim_\mathbb{Q} k_i(K_X + \Delta_i)$, where $\Delta_i \geq 0$ and $k_i \in \mathbb{Q}_+$ for every i, the algebra $R(X, \mathcal{S})$ is an *adjoint ring associated to* $\mathcal{S}$; furthermore, the *adjoint ring associated to the sequence* $D_1, \dots, D_\ell$ is

$$R(X; D_1, \dots, D_\ell) = \bigoplus_{(m_1, \dots, m_\ell) \in \mathbb{N}^\ell} H^0\big(X, \mathcal{O}_X(\lfloor \textstyle\sum m_i D_i \rfloor)\big).$$

Note that then there is a natural projection map $R(X; D_1, \dots, D_\ell) \longrightarrow R(X, \mathcal{S})$.

If $\mathcal{C} \subseteq \operatorname{Div}_\mathbb{R}(X)$ is a rational polyhedral cone, then Lemma 2.11 implies that $\mathcal{S} = \mathcal{C} \cap \operatorname{Div}(X)$ is a finitely generated monoid, and we define the algebra $R(X, \mathcal{C})$, an *adjoint ring associated to* $\mathcal{C}$, to be $R(X, \mathcal{S})$.

Definition 2.23. Let $(X, S + D)$ be a projective pair, where X is smooth, S is a smooth prime divisor and $D \geq 0$ is integral, and fix $\eta \in H^0(X, \mathcal{O}_X(S))$ such that $\operatorname{div} \eta = S$. From the exact sequence

$$0 \longrightarrow H^0(X, \mathcal{O}_X(D - S)) \xrightarrow{\cdot \eta} H^0(X, \mathcal{O}_X(D)) \xrightarrow{\rho_S} H^0(S, \mathcal{O}_S(D))$$

we define $\operatorname{res}_S H^0(X, \mathcal{O}_X(D)) = \operatorname{Im}(\rho_S)$, and for $\sigma \in H^0(X, \mathcal{O}_X(D))$, denote $\sigma_{|S} = \rho_S(\sigma)$. Note that

$$\operatorname{Ker}(\rho_S) = H^0(X, \mathcal{O}_X(D - S)) \cdot \eta,$$

and that $\operatorname{res}_S H^0(X, \mathcal{O}_X(D)) = 0$ if and only if $S \subseteq \operatorname{Bs}|D|$.

If $\mathcal{S} \subseteq \mathrm{Div}_{\mathbb{Q}}(X)$ is a monoid generated by divisors $D_1, \ldots, D_\ell$, the *restriction of* $R(X, \mathcal{S})$ *to* S is the $\mathcal{S}$-graded ring

$$\mathrm{res}_S\, R(X, \mathcal{S}) = \bigoplus_{D \in \mathcal{S}} \mathrm{res}_S\, H^0\big(X, \mathcal{O}_X(\lfloor D \rfloor)\big),$$

and similarly for $\mathrm{res}_S\, R(X; D_1, \ldots, D_\ell)$.

Definition 2.24. Let $\mathcal{S} \subseteq \mathbb{Z}^r$ be a finitely generated monoid and let $R = \bigoplus_{s \in \mathcal{S}} R_s$ be an $\mathcal{S}$-graded algebra. If $\mathcal{S}' \subseteq \mathcal{S}$ is a finitely generated submonoid, then $R' = \bigoplus_{s \in \mathcal{S}'} R_s$ is a *Veronese subring* of R. If there exists a subgroup $\mathbb{L} \subset \mathbb{Z}^r$ of finite index such that $\mathcal{S}' = \mathcal{S} \cap \mathbb{L}$, then R' is a *Veronese subring of finite index* of R.

Lemma 2.25. *Let $\mathcal{S} \subseteq \mathbb{Z}^r$ be a finitely generated monoid and let $R = \bigoplus_{s \in \mathcal{S}} R_s$ be an $\mathcal{S}$-graded algebra. Let $\mathcal{S}' \subseteq \mathcal{S}$ be a finitely generated submonoid and let $R' = \bigoplus_{s \in \mathcal{S}'} R_s$.*

(i) If R is finitely generated over R_0, then R' is finitely generated over R_0.

(ii) If R_0 is Noetherian, R' is a Veronese subring of finite index of R, and R' is finitely generated over R_0, then R is finitely generated over R_0.

Proof. See [ADHL10, Proposition 1.2.2, Proposition 1.2.4]. □

Corollary 2.26. *Let $f\colon Y \longrightarrow X$ be a birational map between smooth projective varieties. Let $D_1, \ldots, D_\ell \in \mathrm{Div}_{\mathbb{Q}}(X)$ and $D'_1, \ldots, D'_\ell \in \mathrm{Div}_{\mathbb{Q}}(Y)$, and assume that there exist positive rational numbers r_i and f-exceptional $\mathbb{Q}$-divisors $E_i \geq 0$ such that $D'_i \sim_{\mathbb{Q}} r_i f^* D_i + E_i$ for every i. Let S be a smooth prime divisor on X and let $T = f_*^{-1} S$.*

Then the ring $R = R(X; D_1, \ldots, D_\ell)$ is finitely generated if and only if the ring $R' = R(Y; D'_1, \ldots, D'_\ell)$ is finitely generated, and the ring $\mathrm{res}_S\, R$ is finitely generated if and only if the ring $\mathrm{res}_T\, R'$ is finitely generated.

Proof. Let k be a positive integer such that all kD_i, $kr_iD'_i$ and kE_i are integral, and such that $kD'_i \sim kr_i f^* D_i + kE_i$ for all i. Then the rings $R(X; kr_1D_1, \ldots, kr_\ell D_\ell)$ and $R(Y; kD'_1, \ldots, kD'_\ell)$ are Veronese subrings of finite index of R and R', respectively, and they are both isomorphic to $R(Y; kr_1 f^* D_1 + kE_1, \ldots, kr_\ell f^* D_\ell + kE_\ell)$. We conclude by Lemma 2.25. The same argument works for restricted rings. □

Lemma 2.27. *Let X be a smooth projective variety, let $D_1, \ldots, D_\ell \in \mathrm{Div}_{\mathbb{Q}}(X)$, and denote $\mathcal{C} = \sum_{i=1}^{\ell} \mathbb{R}_+ D_i \subseteq \mathrm{Div}_{\mathbb{R}}(X)$.*

(i) If $R(X, \mathcal{C})$ is finitely generated, then $R(X; D_1, \ldots, D_\ell)$ is finitely generated.

(ii) Let S be a smooth prime divisor on X. If $\mathrm{res}_S\, R(X, \mathcal{C})$ is finitely generated, then $\mathrm{res}_S\, R(X; D_1, \ldots, D_\ell)$ is finitely generated.

Proof. We only show (i), since (ii) is analogous. Let k be a positive integer such that $D'_i = kD_i \in \mathrm{Div}(X)$ for all i. The monoid $\mathcal{S} = \sum_{i=1}^{\ell} \mathbb{N} D'_i \subseteq \mathrm{Div}(X)$ is a submonoid of $\mathcal{C} \cap \mathrm{Div}(X)$, and thus $R(X, \mathcal{S})$ is finitely generated by Lemma 2.25(i). But then

$R(X;D_1',\dots,D_\ell')$ is finitely generated by [ADHL10, Proposition 1.2.6], and finally $R(X;D_1,\dots,D_\ell)$ is finitely generated by Lemma 2.25(ii). $\square$

A stronger version of the following result can be found in [ELM$^+$06], see [CL10, Theorem 3.5]. Here we prove it as a consequence of Lemma 2.13.

Lemma 2.28. *Let X be a smooth projective variety and let $D_1,\dots,D_\ell\in\operatorname{Div}_{\mathbb{Q}}(X)$ be such that $|D_i|_{\mathbb{Q}}\neq\emptyset$ for each i. Let $V\subseteq\operatorname{Div}_{\mathbb{R}}(X)$ be the subspace spanned by the components of $D_1,\dots,D_\ell$, and let $\mathcal{P}\subseteq V$ be the convex hull of $D_1,\dots,D_\ell$. Assume that the ring $R(X;D_1,\dots,D_\ell)$ is finitely generated. Then:*

(i) $\mathbf{Fix}$ *extends to a rational piecewise affine function on $\mathcal{P}$;*
(ii) there exists a positive integer k such that for every $D\in\mathcal{P}$ and every $m\in\mathbb{N}$, if $\frac{m}{k}D\in\operatorname{Div}(X)$, then $\mathbf{Fix}(D)=\frac{1}{m}\operatorname{Fix}|mD|$.

Proof. Pick a prime divisor $S\in\operatorname{Div}(X)\setminus V$ and a rational function $\eta\in k(X)$ such that $\operatorname{mult}_S\operatorname{div}\eta=1$. Then, setting $D_i'=D_i+\operatorname{div}\eta\sim_{\mathbb{Q}}D_i$, we have $\operatorname{mult}_S D_i'=1$ and $R(X;D_1,\dots,D_\ell)\simeq R(X;D_1',\dots,D_\ell')$. If $\mathcal{P}'\subseteq\operatorname{Div}_{\mathbb{R}}(X)$ is the convex hull of $D_1',\dots,D_\ell'$, it suffices to prove claims (i) and (ii) on $\mathcal{P}'$. Therefore, after replacing D_i by D_i', we may assume that $\mathcal{P}$ belongs to a rational affine hyperplane which does not contain the origin. Denote $\mathcal{P}_{\mathbb{Q}}=\mathcal{P}\cap\operatorname{Div}_{\mathbb{Q}}(X)$.

Fix a prime divisor $G\in V$. For all $D\in\mathcal{P}_{\mathbb{Q}}$ and all $m\in\mathbb{N}$ sufficiently divisible, let $\varphi_m(D)=\frac{1}{m}\operatorname{mult}_G\operatorname{Fix}|mD|$, and set $\varphi(D)=\operatorname{mult}_G\mathbf{Fix}(D)$. Then, in order to show (i), it suffices to prove that φ is rational piecewise affine.

For every $D\in\mathcal{P}_{\mathbb{Q}}$, the ring $R(X,D)$ is finitely generated by Lemma 2.25(i), and so by [Bou89, III.1.2], there exists a positive integer d such that $R(X,dD)$ is generated by $H^0(X,\mathcal{O}_X(dD))$. Thus

$$\varphi(D)=\varphi_d(D),\quad\text{and in particular}\quad\varphi(D)\in\mathbb{Q}. \tag{1}$$

If $\sigma_1,\dots,\sigma_q$ are generators of $R(X;D_1,\dots,D_\ell)$, then there are $G_i\in\mathcal{P}$ and $m_i\in\mathbb{Q}_+$ such that $\sigma_i\in H^0\big(X,\mathcal{O}_X(\lfloor m_iG_i\rfloor)\big)$. Fix $D\in\mathcal{P}_{\mathbb{Q}}$. Let m be a sufficiently divisible positive integer such that $mD\in\sum\mathbb{N}D_i\cap\operatorname{Div}(X)$, and let $\sigma\in H^0(X,\mathcal{O}_X(mD))$ be such that

$$\varphi_m(D)=\frac{1}{m}\operatorname{mult}_G\operatorname{div}\sigma. \tag{2}$$

Then σ is a polynomial in σ_i, thus there are $\alpha_i\in\mathbb{N}$ such that $mD=\sum\alpha_im_iG_i$ and

$$\operatorname{mult}_G\operatorname{div}\sigma=\sum\alpha_i\operatorname{mult}_G\operatorname{div}\sigma_i. \tag{3}$$

Denote $t_{m,i}=\frac{\alpha_im_i}{m}$, and note that $\operatorname{mult}_G\operatorname{div}\sigma_i\geq\varphi(m_iG_i)=m_i\varphi(G_i)$. Then by (2) and (3) we have

$$D=\sum t_{m,i}G_i\qquad\text{and}\qquad\varphi(D)=\inf_{m\in\mathbb{N}}\varphi_m(D)\geq\inf_{m\in\mathbb{N}}\sum t_{m,i}\varphi(G_i).$$

However, for all $t_i \in \mathbb{Q}_+$ with $D = \sum t_i G_i$, by convexity we have $\sum t_i \varphi(G_i) \geq \varphi(D)$. Therefore

$$\varphi(D) = \inf \sum t_i \varphi(G_i),$$

where the infimum is taken over all $(t_1, \ldots, t_q) \in \mathbb{R}^q_+$ such that $D = \sum t_i G_i$. By compactness, there exists $(r_1, \ldots, r_q) \in \mathbb{R}^q_+$ such that $D = \sum r_i G_i$ and $\varphi(D) = \sum r_i \varphi(G_i)$. Thus, φ is rational piecewise affine by Lemma 2.13, and (i) follows.

Now we show (ii). After decomposing $\mathcal{P}$, we may assume that $\mathbf{Fix}$ is rational linear on $\mathbb{R}_+\mathcal{P}$. By Lemma 2.11, the monoid $\mathcal{S} = \mathbb{R}_+\mathcal{P} \cap \operatorname{Div}(X)$ is finitely generated, and let $F_1, \ldots, F_p$ be its generators. By (1), there exists a positive integer k such that $\mathbf{Fix}(F_i) = \frac{1}{k} \operatorname{Fix} |kF_i|$ for all i. Let $D \in \mathcal{P} \cap \operatorname{Div}_{\mathbb{Q}}(X)$, and let $m, \alpha_i \in \mathbb{N}$ be such that $\frac{m}{k} D = \sum \alpha_i F_i \in \mathcal{S}$. Then by (i) and by convexity we have

$$\sum \alpha_i \mathbf{Fix}(F_i) = \tfrac{m}{k} \mathbf{Fix}(D) \leq \tfrac{1}{k} \operatorname{Fix} |mD| \leq \tfrac{1}{k} \sum \alpha_i \operatorname{Fix} |kF_i| = \sum \alpha_i \mathbf{Fix}(F_i),$$

and hence all inequalities are equalities. This completes the proof. $\square$

The following result will be used in the proof of Theorem 1.1.

Theorem 2.29. *Let (X, Δ) be a projective klt pair of dimension n, where Δ is a $\mathbb{Q}$-divisor. Then there exist a projective klt pair (Y, Γ) of dimension at most n and positive integers p and q such that the divisors $p(K_X + \Delta)$ and $q(K_Y + \Gamma)$ are integral, $K_Y + \Gamma$ is big and*

$$R(X, p(K_X + \Delta)) \simeq R(Y, q(K_Y + \Gamma)).$$

Proof. See [FM00, Theorem 5.2]. $\square$

3. Lifting sections

In this section, we prove a slight generalization of the lifting theorem by Hacon and McKernan [HM10], see Theorem 3.4.

We will need the following easy consequence of Kawamata-Viehweg vanishing:

Lemma 3.1. *Let (X, B) be a log smooth projective pair of dimension n, where B is a $\mathbb{Q}$-divisor such that $\lfloor B \rfloor = 0$. Let A be a nef and big $\mathbb{Q}$-divisor.*

- *(i) Let S be a smooth prime divisor such that $S \not\subseteq \operatorname{Supp} B$. If $G \in \operatorname{Div}(X)$ is such that $G \sim_{\mathbb{Q}} K_X + S + A + B$, then $|G_{|S}| = |G|_S$.*
- *(ii) Let $f\colon X \longrightarrow Y$ be a birational morphism to a projective variety Y, and let $U \subseteq X$ be an open set such that $f_{|U}$ is an isomorphism and U intersects at most one irreducible component of B. Let H' be a very ample divisor on Y and let $H = f^*H'$. If $F \in \operatorname{Div}(X)$ is such that $F \sim_{\mathbb{Q}} K_X + (n+1)H + A + B$, then $|F|$ is basepoint free at every point of U.*

Proof. Considering the exact sequence

$$0 \longrightarrow \mathcal{O}_X(G - S) \longrightarrow \mathcal{O}_X(G) \longrightarrow \mathcal{O}_S(G) \longrightarrow 0,$$

Kawamata-Viehweg vanishing implies $H^1(X, \mathcal{O}_X(G - S)) = 0$. In particular, the map $H^0(X, \mathcal{O}_X(G)) \longrightarrow H^0(S, \mathcal{O}_S(G))$ is surjective. This proves (i).

We prove (ii) by induction on n. Let $x \in U$ be a closed point, and pick a general element $T \in |H|$ which contains x. Then by the assumptions on U, it follows that $(X, T + B)$ is log smooth, and since $F_{|T} \sim_{\mathbb{Q}} K_T + nH_{|T} + A_{|T} + B_{|T}$, by induction $F_{|T}$ is free at x. Considering the exact sequence

$$0 \longrightarrow \mathcal{O}_X(F - T) \longrightarrow \mathcal{O}_X(F) \longrightarrow \mathcal{O}_T(F) \longrightarrow 0,$$

Kawamata-Viehweg vanishing implies that $H^1(X, \mathcal{O}_X(F - T)) = 0$. In particular, the map $H^0(X, \mathcal{O}_X(F)) \longrightarrow H^0(T, \mathcal{O}_T(F))$ is surjective, and (ii) follows. $\square$

Lemma 3.2. *Let $(X, S + B)$ be a projective pair, where X is smooth, S is a smooth prime divisor and B is a $\mathbb{Q}$-divisor such that $S \not\subseteq \operatorname{Supp} B$. Let A be a nef and big $\mathbb{Q}$-divisor on X. Assume that $D \in \operatorname{Div}(X)$ is such that $D \sim_{\mathbb{Q}} K_X + S + A + B$, and let $\Sigma \in |D_{|S}|$. Let $\Phi \in \operatorname{Div}_{\mathbb{Q}}(S)$ be such that the pair (S, Φ) is klt and $B_{|S} \leq \Sigma + \Phi$.*

Then $\Sigma \in |D|_S$.

Proof. Let $f\colon Y \longrightarrow X$ be a log resolution of the pair $(X, S + B)$, and write $T = f_*^{-1}S$. Then there are $\mathbb{Q}$-divisors $\Gamma \geq 0$ and $E \geq 0$ on Y with no common components such that $T \not\subseteq \operatorname{Supp} \Gamma$, E is f-exceptional, and

$$K_Y + T + \Gamma = f^*(K_X + S + B) + E.$$

Let $C = \Gamma - E$ and

$$G = f^*D - \lfloor C \rfloor = f^*D - \lfloor \Gamma \rfloor + \lceil E \rceil. \tag{4}$$

Then

$$G - (K_Y + T + \{C\}) \sim_{\mathbb{Q}} f^*(K_X + S + A + B) - (K_Y + T + C) = f^*A$$

is nef and big, and Lemma 3.1(i) implies that

$$|G_{|T}| = |G|_T. \tag{5}$$

Moreover, since $E \geq 0$ is f-exceptional, we have

$$\begin{aligned} |G|_T + \lfloor \Gamma \rfloor_{|T} &= |f^*D - \lfloor \Gamma \rfloor + \lceil E \rceil|_T + \lfloor \Gamma \rfloor_{|T} \\ &\subseteq |f^*D + \lceil E \rceil|_T = |f^*D|_T + \lceil E \rceil_{|T}. \end{aligned} \tag{6}$$

Denote $g = f_{|T}\colon T \longrightarrow S$. Then

$$K_T + C_{|T} = g^*(K_S + B_{|S}) \quad \text{and} \quad K_T + \Psi = g^*(K_S + \Phi),$$

for some $\mathbb{Q}$-divisor Ψ on T, and note that $\lfloor \Psi \rfloor \leq 0$ since (S, Φ) is klt. Therefore

$$g^*(B_{|S} - \Phi) = C_{|T} - \Psi. \tag{7}$$

By assumption we have that $B_{|S} \leq \Sigma + \Phi$, that $g^*\Sigma$ is integral, and that the support of $C + T$ has normal crossings, so this together with (7) gives

$$\begin{aligned} g^*\Sigma \geq g^*\Sigma + \lfloor \Psi \rfloor &= \lfloor g^*\Sigma + \Psi \rfloor \geq \lfloor g^*(B_{|S} - \Phi) + \Psi \rfloor \\ &= \lfloor C_{|T} \rfloor = \lfloor C \rfloor_{|T} = (f^*D)_{|T} - G_{|T}. \end{aligned}$$

Denote

$$R = G_{|T} - (f^*D)_{|T} + g^*\Sigma.$$

Then $R \geq 0$ by the above, and $g^*\Sigma \in |(f^*D)_{|T}|$ implies $R \in |G_{|T}| = |G|_T$ by (5). Therefore $R + \lfloor \Gamma \rfloor_{|T} \in |f^*D|_T + \lceil E \rceil_{|T}$ by (6), and this together with (4) yields

$$g^*\Sigma = R + (f^*D)_{|T} - G_{|T} = R + \lfloor \Gamma \rfloor_{|T} - \lceil E \rceil_{|T} \in |f^*D|_T,$$

hence the claim follows. $\square$

Lemma 3.3. *Let $(X, S + B + D)$ be a log smooth projective pair, where S is a prime divisor, B is a $\mathbb{Q}$-divisor such that $\lfloor B \rfloor = 0$ and $S \not\subseteq \operatorname{Supp} B$, and $D \geq 0$ is a $\mathbb{Q}$-divisor such that D and $S + B$ have no common components. Let P be a nef $\mathbb{Q}$-divisor and denote $\Delta = S + B + P$. Assume that*

$$K_X + \Delta \sim_{\mathbb{Q}} D.$$

Let k be a positive integer such that kP and kB are integral, and write $\Omega = (B+P)_{|S}$.

Then there is a very ample divisor H such that for all divisors $\Sigma \in |k(K_S + \Omega)|$ and $U \in |H_{|S}|$, and for every positive integer l we have

$$l\Sigma + U \in |lk(K_X + \Delta) + H|_S.$$

Proof. For any $m \geq 0$, let $l_m = \lfloor \frac{m}{k} \rfloor$ and $r_m = m - l_m k \in \{0, 1, \ldots, k-1\}$, define $B_m = \lceil mB \rceil - \lceil (m-1)B \rceil$, and set $P_m = kP$ if $r_m = 0$, and otherwise $P_m = 0$. Let

$$D_m = \sum_{i=1}^{m} (K_X + S + P_i + B_i) = m(K_X + S) + l_m kP + \lceil mB \rceil,$$

and note that D_m is integral and

$$D_m = l_m k(K_X + \Delta) + D_{r_m}. \tag{8}$$

By Serre vanishing, we can pick a very ample divisor H on X such that:

(i) $D_j + H$ is ample and basepoint free for every $0 \leq j \leq k-1$,
(ii) $|D_k + H|_S = |(D_k + H)_{|S}|$.

We claim that for all divisors $\Sigma \in |k(K_S + \Omega)|$ and $U_m \in |(D_{r_m} + H)_{|S}|$ we have

$$l_m\Sigma + U_m \in |D_m + H|_S.$$

The case $r_m = 0$ immediately implies the lemma.

We prove the claim by induction on m. The case $m = k$ is covered by (ii). Now let $m > k$, and pick a rational number $0 < \delta \ll 1$ such that $D_{r_{m-1}} + H + \delta B_m$ is ample. Note that $0 \leq B_m \leq \lceil B \rceil$, that $(X, S + B + D)$ is log smooth, and that

D and $S+B$ have no common components. Thus, there exists a rational number $0<\varepsilon\ll 1$ such that, if we define

$$F=(1-\varepsilon\delta)B_m+l_{m-1}k\varepsilon D, \tag{9}$$

then $(X,S+F)$ is log smooth, $\lfloor F\rfloor=0$ and $S\not\subseteq\operatorname{Supp}F$. In particular, if W is a general element of the free linear system $|(D_{r_{m-1}}+H)_{|S}|$ and

$$\Phi=F_{|S}+(1-\varepsilon)W, \tag{10}$$

then (S,Φ) is klt.

By induction, there is a divisor $\Upsilon\in|D_{m-1}+H|$ such that $S\not\subseteq\operatorname{Supp}\Upsilon$ and

$$\Upsilon_{|S}=l_{m-1}\Sigma+W.$$

Denoting $C=(1-\varepsilon)\Upsilon+F$, by (9) we have

$$C\sim_{\mathbb{Q}}(1-\varepsilon)(D_{m-1}+H)+(1-\varepsilon\delta)B_m+l_{m-1}k\varepsilon D, \tag{11}$$

and (10) yields

$$C_{|S}=(1-\varepsilon)\Upsilon_{|S}+F_{|S}\leq l_{m-1}\Sigma+\Phi\leq(l_m\Sigma+U_m)+\Phi. \tag{12}$$

By the choice of δ and since P_m is nef, the $\mathbb{Q}$-divisor

$$A=\varepsilon(D_{r_{m-1}}+H+\delta B_m)+P_m \tag{13}$$

is ample. Then by (8), (13) and (11) we have

$$\begin{aligned}D_m+H&=K_X+S+D_{m-1}+B_m+P_m+H\\&=K_X+S+(1-\varepsilon)D_{m-1}+l_{m-1}k\varepsilon(K_X+\Delta)+\varepsilon D_{r_{m-1}}+B_m+P_m+H\\&\sim_{\mathbb{Q}}K_X+S+A+(1-\varepsilon)D_{m-1}+l_{m-1}k\varepsilon D+(1-\varepsilon\delta)B_m+(1-\varepsilon)H\\&\sim_{\mathbb{Q}}K_X+S+A+C,\end{aligned}$$

and thus $l_m\Sigma+U_m\in|D_m+H|_S$ by (12) and Lemma 3.2. $\square$

Theorem 3.4. *Let $(X,S+B)$ be a log smooth projective pair, where S is a prime divisor, and B is a $\mathbb{Q}$-divisor such that $S\not\subseteq\operatorname{Supp}B$ and $\lfloor B\rfloor=0$. Let A be an ample $\mathbb{Q}$-divisor on X and denote $\Delta=S+A+B$. Let $C\geq 0$ be a $\mathbb{Q}$-divisor on S such that (S,C) is canonical, and let m be a positive integer such that mA, mB and mC are integral.*

Assume that there exists a positive integer $q\gg 0$ such that qA is very ample, $S\not\subseteq\operatorname{Bs}|qm(K_X+\Delta+\frac{1}{m}A)|$ and

$$C\leq B_{|S}-B_{|S}\wedge\frac{1}{qm}\operatorname{Fix}|qm(K_X+\Delta+\tfrac{1}{m}A)|_S.$$

Then

$$|m(K_S+A_{|S}+C)|+m(B_{|S}-C)\subseteq|m(K_X+\Delta)|_S.$$

In particular, if $|m(K_S+A_{|S}+C)|\neq\emptyset$, then $|m(K_X+\Delta)|_S\neq\emptyset$, and

$$\operatorname{Fix}|m(K_S+A_{|S}+C)|+m(B_{|S}-C)\geq\operatorname{Fix}|m(K_X+\Delta)|_S\geq m\,\mathbf{Fix}_S(K_X+\Delta).$$

Proof. Let $f\colon Y \longrightarrow X$ be a log resolution of the pair $(X, S+B)$ and of the linear system $|qm(K_X + \Delta + \frac{1}{m}A)|$, and write $T = f_*^{-1}S$. Then there are $\mathbb{Q}$-divisors $B', E \geq 0$ on Y with no common components, such that E is f-exceptional and

$$K_Y + T + B' = f^*(K_X + S + B) + E.$$

Note that

$$K_T + B'_{|T} = g^*(K_S + B_{|S}) + E_{|T},$$

and since $(Y, T + B' + E)$ is log smooth and B' and E do not have common components, it follows that $B'_{|T}$ and $E_{|T}$ do not have common components, and in particular, $E_{|T}$ is g-exceptional and $g_*B'_{|T} = B_{|S}$. Let $\Gamma = T + f^*A + B'$, and define

$$F_q = \tfrac{1}{qm}\operatorname{Fix}|qm(K_Y + \Gamma + \tfrac{1}{m}f^*A)|, \quad B'_q = B' - B' \wedge F_q, \quad \Gamma_q = T + B'_q + f^*A.$$

Since $(Y, T + B' + F_q)$ is log smooth, $\operatorname{Mob}\big(qm(K_Y + \Gamma + \frac{1}{m}f^*A)\big)$ is basepoint free, and $T \not\subseteq \mathbf{B}(K_Y + \Gamma + \frac{1}{m}f^*A)$, by Bertini's theorem there exists a $\mathbb{Q}$-divisor $D \geq 0$ such that

$$K_Y + \Gamma_q + \tfrac{1}{m}f^*A \sim_{\mathbb{Q}} D,$$

the pair $(Y, T + B'_q + D)$ is log smooth, and D does not contain any component of $T + B'_q$. Let $g = f_{|T}\colon T \longrightarrow S$. Since (S, C) is canonical, there is a g-exceptional $\mathbb{Q}$-divisor $F \geq 0$ on T such that

$$K_T + C' = g^*(K_S + C) + F,$$

where $C' = g_*^{-1}C$. We claim that $C' \leq B'_{q|T}$. Assuming the claim, let us show how it implies the theorem.

By Lemma 3.3, there exists a very ample divisor H on Y such that for all divisors $\Sigma' \in |qm(K_T + (B'_q + (1 + \frac{1}{m})f^*A)_{|T})|$ and $U \in |H_{|T}|$, and for every positive integer p we have

$$p\Sigma' + U \in |pqm(K_Y + \Gamma_q + \tfrac{1}{m}f^*A) + H|_T.$$

Pick an f-exceptional $\mathbb{Q}$-divisor $G \geq 0$ such that $\lfloor B' + \frac{1}{m}G \rfloor = 0$ and $f^*A - G$ is ample. In particular, $(T, (B' + \frac{1}{m}G)_{|T})$ is klt. Let $W_1 \in |q(f^*A)_{|T}|$ and $W_2 \in |H_{|T}|$ be general sections. Pick a positive integer $k \gg 0$ such that, if we denote $l = kq$, $W = kW_1 + W_2$ and $\Phi = B'_{|T} + \frac{1}{m}G_{|T} + \frac{1}{l}W$, then the $\mathbb{Q}$-divisor

$$A_0 = \frac{1}{m}(f^*A - G) - \frac{m-1}{ml}H \tag{14}$$

is ample and the pair (T, Φ) is klt.

Fix $\Sigma \in |m(K_S + A_{|S} + C)|$. Since $C' \leq B'_{q|T}$ by the claim, it is easy to check that

$$qg^*\Sigma + qm(F + B'_{q|T} - C') + W_1 \in |qm(K_T + (B'_q + (1 + \tfrac{1}{m})f^*A)_{|T})|.$$

Then, by the choice of H, there exists $\Upsilon \in |lm(K_Y + \Gamma_q + \frac{1}{m}f^*A) + H|$ such that $T \not\subseteq \operatorname{Supp}\Upsilon$ and

$$\Upsilon_{|T} = lg^*\Sigma + lm(F + B'_{q|T} - C') + W.$$

Denoting

$$B_0 = \frac{m-1}{ml}\Upsilon + (m-1)(\Gamma - \Gamma_q) + B' + \frac{1}{m}G, \tag{15}$$

relations (14) and (15) imply

$$\begin{aligned} m(K_Y+\Gamma) &= K_Y + T + (m-1)(K_Y + \Gamma + \tfrac{1}{m}f^*A) + \tfrac{1}{m}f^*A + B' \\ &\sim_{\mathbb{Q}} K_Y + T + \tfrac{m-1}{ml}\Upsilon + (m-1)(\Gamma-\Gamma_q) + \tfrac{1}{m}f^*A - \tfrac{m-1}{ml}H + B' \\ &= K_Y + T + A_0 + B_0. \end{aligned} \tag{16}$$

Noting that $\Gamma - \Gamma_q = B' - B'_q$, we have

$$\begin{aligned} B_{0|T} = \tfrac{m-1}{m}g^*\Sigma + (m-1)\big(F + B'_{q|T} - C' + (\Gamma-\Gamma_q)_{|T}\big) \\ + \tfrac{m-1}{ml}W + B'_{|T} + \tfrac{1}{m}G_{|T} \leq g^*\Sigma + m(F + B'_{|T} - C') + \Phi, \end{aligned} \tag{17}$$

and since $g^*\Sigma + m(F + B'_{|T} - C') \in |m(K_Y+\Gamma)_{|T}|$, by (16), (17) and Lemma 3.2 we obtain

$$g^*\Sigma + m(F + B'_{|T} - C') \in |m(K_Y+\Gamma)|_T.$$

Pushing forward by g yields $\Sigma + m(B_{|S} - C) \in |m(K_X+\Delta)|_S$ and the lemma follows.

Now we prove the claim stated above. Since $\operatorname{Mob}\big(qm(K_Y + \Gamma + \frac{1}{m}f^*A)\big)$ is basepoint free and T is not a component of F_q, it follows that $\frac{1}{qm}\operatorname{Fix}|qm(K_Y+\Gamma+\frac{1}{m}f^*A)|_T = F_{q|T}$ and

$$B'_{q|T} = B'_{|T} - (B' \wedge F_q)_{|T} = B'_{|T} - B'_{|T} \wedge \tfrac{1}{qm}\operatorname{Fix}|qm(K_Y+\Gamma+\tfrac{1}{m}f^*A)|_T.$$

Furthermore, we have

$$g_*\operatorname{Fix}|qm(K_Y+\Gamma+\tfrac{1}{m}f^*A)|_T = \operatorname{Fix}|qm(K_X+\Delta+\tfrac{1}{m}A)|_S,$$

so

$$g_*C' = C \leq B_{|S} - B_{|S} \wedge \tfrac{1}{qm}\operatorname{Fix}|qm(K_X+\Delta+\tfrac{1}{m}A)|_S = g_*B'_{q|T}.$$

Therefore $C' \leq B'_{q|T}$, since $B'_{q|T} \geq 0$ and $C' = g_*^{-1}C$. $\square$

We immediately obtain the lifting theorem from [HM10].

Corollary 3.5. *Let $(X, S+B)$ be a log smooth projective pair, where S is a prime divisor, and B is a $\mathbb{Q}$-divisor such that $S \not\subseteq \operatorname{Supp} B$, $\lfloor B \rfloor = 0$ and $(S, B_{|S})$ is canonical. Let A be an ample $\mathbb{Q}$-divisor on X and denote $\Delta = S + A + B$. Let m be a positive integer such that mA and mB are integral, and such that $S \not\subseteq \operatorname{Bs}|m(K_X+\Delta)|$. Denote $\Phi_m = B_{|S} - B_{|S} \wedge \frac{1}{m}\operatorname{Fix}|m(K_X+\Delta)|_S$.*

Then

$$|m(K_S + A_{|S} + \Phi_m)| + m(B_{|S} - \Phi_m) = |m(K_X+\Delta)|_S.$$

Proof. Since $\Phi_m \leq B_{|S} - B_{|S} \wedge \frac{1}{qm}\operatorname{Fix}|qm(K_X+\Delta+\frac{1}{m}A)|_S$ for any positive integer q, the inclusion $|m(K_S + A_{|S} + \Phi_m)| + m(B_{|S} - \Phi_m) \subseteq |m(K_X+\Delta)|_S$ follows from Theorem 3.4, whereas the reverse inclusion is implied by $m(B_{|S} - \Phi_m) \leq \operatorname{Fix}|m(K_X+\Delta)|_S$. $\square$

Lemma 3.6. *Let X be a smooth projective variety and let S be a smooth prime divisor on X. Let D be a $\mathbb{Q}$-divisor such that $S \not\subseteq \mathbf{B}(D)$, and let A be an ample $\mathbb{Q}$-divisor. Then*

$$\frac{1}{q} \operatorname{Fix} |q(D+A)|_S \leq \mathbf{Fix}_S(D)$$

for any sufficiently divisible positive integer q.

Proof. Let P be a prime divisor on S and let $\gamma = \operatorname{mult}_P \mathbf{Fix}_S(D)$. It is enough to show that

$$\operatorname{mult}_P \frac{1}{q} \operatorname{Fix} |q(D+A)|_S \leq \gamma$$

for some sufficiently divisible positive integer q.

Assume first that $\gamma > 0$. Let $\varepsilon > 0$ be a rational number such that $\varepsilon D + A$ is ample, and pick a positive integer m such that

$$\frac{1-\varepsilon}{m} \operatorname{mult}_P \operatorname{Fix} |mD|_S \leq \gamma.$$

Let q be a sufficiently divisible positive integer such that the divisor $q(\varepsilon D + A)$ is very ample, and such that m divides $q(1-\varepsilon)$. Then

$$\begin{aligned}\frac{1}{q} \operatorname{mult}_P \operatorname{Fix} |q(D+A)|_S &= \frac{1}{q} \operatorname{mult}_P \operatorname{Fix} |q(1-\varepsilon)D + q(\varepsilon D + A)|_S \\ &\leq \frac{1}{q} \operatorname{mult}_P \operatorname{Fix} |q(1-\varepsilon)D|_S \leq \frac{1-\varepsilon}{m} \operatorname{mult}_P \operatorname{Fix} |mD|_S \leq \gamma.\end{aligned}$$

Now assume that $\gamma = 0$. Let $n = \dim X$ and let H be a very ample divisor on X. Pick a positive integer q such that qA and qD are integral, and such that

$$C = qA - K_X - S - nH \tag{18}$$

is ample. Then there exists a $\mathbb{Q}$-divisor $D' \geq 0$ such that $D' \sim_{\mathbb{Q}} D$, $S \not\subseteq \operatorname{Supp} D'$ and $\operatorname{mult}_P(D'_{|S}) < \frac{1}{q}$. Let $f\colon Y \longrightarrow X$ be a log resolution of $(X, S + D')$ which is obtained as a sequence of blowups along smooth centres. Let $T = f_*^{-1}S$, and let $E \geq 0$ be the f-exceptional integral divisor such that

$$K_Y + T = f^*(K_X + S) + E.$$

Then, denoting $F = qf^*(D+A) - \lfloor qf^*D' \rfloor + E$, by (18) we have

$$F \sim_{\mathbb{Q}} qf^*A + \{qf^*D'\} + E = K_Y + T + f^*(nH + C) + \{qf^*D'\},$$

and in particular $|F_{|T}| = |F|_T$ by Lemma 3.1(i). Denote $g = f_{|T}\colon T \longrightarrow S$ and let $P' = g_*^{-1}P$. Since $F_{|T} \sim_{\mathbb{Q}} K_T + g^*(nH_{|S}) + g^*(C_{|S}) + \{qf^*D'\}_{|T}$ and g is an isomorphism at the generic point of P', Lemma 3.1(ii) implies that the base locus of $|F_{|T}|$ does not contain P'. In particular, if $V \in |F|$ is a general element, then $P \not\subseteq \operatorname{Supp} f_*V$.

Let $U = V + \lfloor qf^*D' \rfloor \in |qf^*(D+A)+E|$. Since E is f-exceptional, this implies that $f_*U \in |q(D+A)|$, and since $f_*\lfloor qf^*D' \rfloor \leq qD'$, we have

$$\operatorname{mult}_P(f_*U)_{|S} = \operatorname{mult}_P(f_*V)_{|S} + \operatorname{mult}_P(f_*\lfloor qf^*D' \rfloor)_{|S} \leq \operatorname{mult}_P qD'_{|S} < 1.$$

Thus, $\operatorname{mult}_P(f_*U)_{|S} = 0$ and the lemma follows. □

4. $\mathcal{B}^S_A(V)$ is a rational polytope

In this section, we prove several results which will be used in Sections 5 and 6 to deduce the non-vanishing theorem and the finite generation of the restricted ring.

We introduce a function $\mathbf{\Phi}$ which is naturally related to the lifting theorem 3.4. More precisely, with the same notation as in Setup 4.1, given a $\mathbb{Q}$-divisor $B \in \mathcal{B}^S_A(V)$, a sufficiently divisible positive integer m and a section $\Sigma \in |m(K_S + A_{|S} + \mathbf{\Phi}(B))|$, we can lift $\Sigma + m(B_{|S} - \mathbf{\Phi}(B))$ to X as a section of $|m(K_X + S + A + B)|$. Using Diophantine approximation we prove that $\mathcal{B}^S_A(V)$ is a rational polytope and that, modulo some additional technical assumptions, the function $\mathbf{\Phi}(B)$ is rational piecewise linear. This latter fact implies that the restricted ring is finitely generated: it shows that the ring in question is in fact an adjoint ring on a variety of lower dimension, thus we are able to apply induction, see Lemma 6.2.

In all results of this section we work in the following setup, and we write "Setup 4.1$_n$" to denote "Setup 4.1 in dimension n."

Setup 4.1. We assume Theorem A$_{n-1}$ and Theorem B$_{n-1}$. Let $(X, S + \sum_{i=1}^p S_i)$ be a log smooth projective pair of dimension n, where S and all S_i are distinct prime divisors. Let $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \operatorname{Div}_{\mathbb{R}}(X)$, let A be an ample $\mathbb{Q}$-divisor on X, and let $W \subseteq \operatorname{Div}_{\mathbb{R}}(S)$ be the subspace spanned by the components of $\sum S_{i|S}$.

The set $\mathcal{E}_{A_{|S}}(W)$ is a rational polytope by Theorem B$_{n-1}$. If $E_1, \dots, E_d$ are its extreme points, the ring $R(S; K_S + A_{|S} + E_1, \dots, K_S + A_{|S} + E_d)$ is finitely generated by Theorem A$_{n-1}$. Therefore, if we set

$$\mathbf{F}(E) = \mathbf{Fix}(K_S + A_{|S} + E)$$

for a $\mathbb{Q}$-divisor $E \in \mathcal{E}_{A_{|S}}(W)$, then Lemma 2.28 implies that $\mathbf{F}$ extends to a rational piecewise affine function on $\mathcal{E}_{A_{|S}}(W)$, and there exists a positive integer k with the property that

$$\mathbf{F}(E) = \frac{1}{m} \operatorname{Fix} |m(K_S + A_{|S} + E)| \tag{19}$$

for every $E \in \mathcal{E}_{A_{|S}}(W)$ and every $m \in \mathbb{N}$ such that mA/k and mE/k are integral. We define the set

$$\mathcal{F} = \{E \in \mathcal{E}_{A_{|S}}(W) \mid E \wedge \mathbf{F}(E) = 0\}.$$

Then $\mathcal{F}$ is a subset of $\mathcal{E}_{A_{|S}}(W)$ defined by finitely many linear equalities and inequalities. Thus, there are finitely many rational polytopes $\mathcal{F}_i$ such that $\mathcal{F} = \bigcup_i \mathcal{F}_i$.

For a $\mathbb{Q}$-divisor $B \in \mathcal{B}_A^S(V)$, set

$$\mathbf{F}_S(B) = \mathbf{Fix}_S(K_X + S + A + B),$$

and for every positive integer m such that mA, mB are integral and $S \not\subseteq \operatorname{Bs}|m(K_X + S + A + B)|$, denote

$$\Phi_m(B) = B_{|S} - B_{|S} \wedge \tfrac{1}{m} \operatorname{Fix}|m(K_X + S + A + B)|_S.$$

Let $\mathbf{\Phi}(B) = B_{|S} - B_{|S} \wedge \mathbf{F}_S(B)$, and note that $\mathbf{\Phi}(B) = \limsup \Phi_m(B)$.

Lemma 4.2. *Let the assumptions of Setup 4.1$_n$ hold. If $B \in \mathcal{B}_A^S(V)$, then $\Phi_m(B) \in \mathcal{E}_{A_{|S}}(W)$ and $\Phi_m(B) \wedge \mathbf{F}(\Phi_m(B)) = 0$. In particular, if $\mathcal{B}_A^S(V) \neq \emptyset$, then $\mathcal{F} \neq \emptyset$.*

Proof. Clearly $\Phi_m(B) \in \mathcal{E}_{A_{|S}}(W)$. For the second claim, note that since $m(B_{|S} - \Phi_m(B)) \leq \operatorname{Fix}|m(K_X + S + A + B)|_S$, we have

$$|m(K_S + A_{|S} + \Phi_m(B))| + m(B_{|S} - \Phi_m(B)) \supseteq |m(K_X + S + A + B)|_S,$$

so

$$\operatorname{Fix}|m(K_S + A_{|S} + \Phi_m(B))| + m(B_{|S} - \Phi_m(B)) \leq \operatorname{Fix}|m(K_X + S + A + B)|_S. \tag{20}$$

If T is a component of $\Phi_m(B)$, then by definition

$$\operatorname{mult}_T \Phi_m(B) = \operatorname{mult}_T B_{|S} - \tfrac{1}{m} \operatorname{mult}_T \operatorname{Fix}|m(K_X + S + A + B)|_S,$$

which together with (20) gives $\operatorname{mult}_T \operatorname{Fix}|m(K_S + A_{|S} + \Phi_m(B))| = 0$. Hence $\operatorname{mult}_T \operatorname{Fix}|km(K_S + A_{|S} + \Phi_m(B))| = 0$ for every $k \in \mathbb{N}$, which implies

$$\Phi_m(B) \wedge \tfrac{1}{km} \operatorname{Fix}|km(K_S + A_{|S} + \Phi_m(B))| = 0.$$

Letting $k \longrightarrow \infty$ yields the lemma. $\square$

The main result of this section is:

Theorem 4.3. *Let the assumptions of Setup 4.1$_n$ hold. Let $\mathcal{G}$ be a rational polytope contained in the interior of $\mathcal{L}(V)$, and assume that $(S, G_{|S})$ is terminal for every $G \in \mathcal{G}$. Denote $\mathcal{P} = \mathcal{G} \cap \mathcal{B}_A^S(V)$. Then*

(i) $\mathcal{P}$ is a rational polytope,

(ii) $\mathbf{\Phi}$ extends to a rational piecewise affine function on $\mathcal{P}$, and there exists a positive integer ℓ with the property that $\mathbf{\Phi}(P) = \Phi_m(P)$ for every $P \in \mathcal{P}$ and every positive integer m such that mP/ℓ is integral.

We describe briefly the strategy of the proof. The goal of the construction is to show that the *subgraph* of $\mathbf{\Phi}$ is a finite union of *convex* rational polytopes, which in itself does not have to be convex. Indeed, the function $B_{|S} \wedge \mathbf{F}_S(B)$ is not a convex function since it is defined as the minimum of two convex functions. This is one of the technical obstacles in the proof of Theorem 4.3, and it is addressed in Step 3. The main point there is to show that the locus where it *is* convex is a rational polytope. This requires working in the space $\operatorname{Div}_{\mathbb{R}}(X) \times \operatorname{Div}_{\mathbb{R}}(S)$, and it is essentially dealt with in Lemma 4.4. Then part (i) of Theorem 4.3 follows immediately by projecting this subgraph onto $\mathrm{Div}_{\mathbb{R}}(X)$.

The fact that $\mathcal{B}_A^S(V)$ is a rational polytope is an easy, but technical consequence. The details are discussed in Corollary 4.6.

Lemma 4.4. *Let the assumptions of Setup 4.1$_n$ hold. Let $\mathcal{G}$ be a rational polytope contained in the interior of $\mathcal{L}(V)$, and assume that $(S, G_{|S})$ is terminal for every $G \in \mathcal{G}$. Fix a rational polytope $\mathcal{F}_i$ in the decomposition $\mathcal{F} = \bigcup_i \mathcal{F}_i$ and let*

$$\mathcal{Q}_i' = \{(G, F) \in \mathrm{Div}_{\mathbb{Q}}(X) \times \mathrm{Div}_{\mathbb{Q}}(S) \mid G \in \mathcal{G} \cap \mathcal{B}_A^S(V), F \in \mathcal{F}_i, F \leq \mathbf{\Phi}(G)\}.$$

Then the convex hull of $\mathcal{Q}_i'$ is a rational polytope.

Proof. Step 1. Let $\mathcal{Q}_i$ be the convex hull of $\mathcal{Q}_i'$. We first prove that $\mathcal{Q}_i'$ is dense in $\mathcal{Q}_i$.

To this end, fix $(G_0, F_0), (G_1, F_1) \in \mathcal{Q}_i'$, and for a rational number $0 \leq t \leq 1$ set

$$G_t = (1-t)G_0 + tG_1 \in \mathcal{P} \quad \text{and} \quad F_t = (1-t)F_0 + tF_1 \in \mathcal{F}_i.$$

It suffices to show that $(G_t, F_t) \in \mathcal{Q}_i'$, i.e. that $F_t \leq \mathbf{\Phi}(G_t)$ for every t.

Let T be a prime divisor in W. If $\operatorname{mult}_T F_t = 0$ for some $0 < t < 1$, then since $\operatorname{mult}_T F_0 \geq 0$ and $\operatorname{mult}_T F_1 \geq 0$ we must have $\operatorname{mult}_T F_t = 0$ for all rational $t \in [0,1]$, and in particular $\operatorname{mult}_T F_t \leq \operatorname{mult}_T \mathbf{\Phi}(G_t)$.

Otherwise, we have $\operatorname{mult}_T F_t > 0$ for all $0 < t < 1$, and it follows from the definition of $\mathcal{F}_i$ and by continuity of $\mathbf{F}$ that

$$\operatorname{mult}_T \mathbf{F}(F_t) = 0 \quad \text{for all} \quad t \in [0,1]. \tag{21}$$

Let m be a positive integer such that mG_j/k and mF_j/k are integral for $j \in \{0,1\}$. By Lemma 3.6, we have $\frac{1}{q}\operatorname{Fix}|q(K_X + S + A + G_j + \frac{1}{m}A)|_S \leq \mathbf{F}_S(G_j)$ for any sufficiently divisible positive integer q. Since $\mathbf{F}(F_j) = \frac{1}{m}\operatorname{Fix}|m(K_S + A_{|S} + F_j)|$ by assumption, Theorem 3.4 implies

$$m\,\mathbf{F}(F_j) + m(G_{j|S} - F_j) \geq m\,\mathbf{F}_S(G_j),$$

and therefore $\operatorname{mult}_T\big(G_{j|S} - \mathbf{F}_S(P_j)\big) \geq \operatorname{mult}_T F_j$ by (21). Hence,

$$\operatorname{mult}_T F_t \leq \operatorname{mult}_T\big(G_{t|S} - \mathbf{F}_S(G_t)\big) \leq \operatorname{mult}_T \mathbf{\Phi}(G_t)$$

for all t by convexity of the function $\mathbf{F}_S$.

Step 2. Let

$$\mathcal{C}_i = \{(G, F) \in \mathcal{G} \times \mathcal{F}_i \mid F \leq G_{|S}\}.$$

Note that $\mathcal{C}_i$ is a rational polytope and $\overline{\mathcal{Q}_i} \subseteq \mathcal{C}_i$. Fix a rational number $0 < \varepsilon \ll 1$ such that $D + \frac{1}{4}A$ is ample for any $D \in V$ with $\|D\| < \varepsilon$, and $\varepsilon(K_X + S + A + B) + \frac{1}{4}A$ is ample for any $B \in \mathcal{L}(V)$. In the next two steps, we prove the following:

Claim 4.5. *Suppose we are given $(B,C) \in \overline{\mathcal{Q}_i}$ and $(\Gamma,\Psi) \in \operatorname{face}\big(\mathcal{C}_i,(B,C)\big)$. Assume that there exist a positive integer m and a rational number $0<\phi\leq 1$ such that mA/k, $m\Gamma/k$ and $m\Psi/k$ are integral, and $\|\Gamma-B\|<\frac{\phi\varepsilon}{2m}$ and $\|\Psi-C\|<\frac{\phi\varepsilon}{2m}$. Assume that for any prime divisor T on S we have*

$$\operatorname{mult}_T(B_{|S}-C)>\phi \qquad \text{or} \qquad \operatorname{mult}_T(B_{|S}-C)\leq \operatorname{mult}_T(\Gamma_{|S}-\Psi).$$

Then $(\Gamma,\Psi)\in\mathcal{Q}_i'$.

Step 3. Since $(B,C)\in\overline{\mathcal{Q}_i}$, and $\mathcal{Q}_i'$ is dense in $\mathcal{Q}_i$ by Step 1, for every $\delta>0$ there exists a point $(B_\delta,C_\delta)\in\mathcal{Q}_i'$ such that $\|B-B_\delta\|<\frac{\delta}{2}$ and $\|C-C_\delta\|<\frac{\delta}{2}$. Let us show that $S\not\subseteq\mathbf{B}(K_X+S+A+\Gamma+\frac{1}{2m}A)$, and that for all prime divisors T on S and for all $0<\delta<\frac{\varepsilon}{m}$, we have

$$\operatorname{mult}_T\mathbf{Fix}_S(K_X+S+A+\Gamma+\tfrac{1}{2m}A)\leq \operatorname{mult}_T\big(\Gamma_{|S}-\Psi\big)+\operatorname{mult}_T\mathbf{F}(C_\delta)+\delta. \tag{22}$$

To this end, note that since $\|\Gamma-B_\delta\|\leq\|\Gamma-B\|+\|B-B_\delta\|\leq\frac{\varepsilon}{m}$, the $\mathbb{Q}$-divisors

$$H=\Gamma-B_\delta+\tfrac{1}{4m}A \qquad \text{and} \qquad G=\tfrac{\varepsilon}{m}(K_X+S+A+B_\delta)+\tfrac{1}{4m}A$$

are ample. By assumption and by Lemma 3.6, there exists a positive integer q such that $S\not\subseteq\operatorname{Bs}|q(K_X+S+A+B_\delta)|$,

$$\tfrac{1}{q}\operatorname{Fix}|q(K_S+A_{|S}+C_\delta)|=\mathbf{F}(C_\delta), \tag{23}$$

and

$$\tfrac{1}{q}\operatorname{Fix}|q(K_X+S+A+B_\delta+H+\tfrac{1}{4m}A)|_S\leq\mathbf{Fix}_S(K_X+S+A+B_\delta+\tfrac{1}{4m}A). \tag{24}$$

By Lemma 3.6, there is an integer $w\gg 0$ such that

$$\tfrac{1}{wq}\operatorname{Fix}|wq(K_X+S+A+B_\delta+\tfrac{1}{q}A)|_S\leq\mathbf{F}_S(B_\delta),$$

so, as $(B_\delta,C_\delta)\in\mathcal{Q}_i'$, we have

$$C_\delta\leq\mathbf{\Phi}(B_\delta)\leq B_{\delta|S}-B_{\delta|S}\wedge\tfrac{1}{wq}\operatorname{Fix}|wq(K_X+S+A+B_\delta+\tfrac{1}{q}A)|_S.$$

Hence Theorem 3.4 and (23) imply

$$\mathbf{F}_S(B_\delta)\leq B_{\delta|S}-C_\delta+\mathbf{F}(C_\delta). \tag{25}$$

As $\Gamma+\frac{1}{2m}A=B_\delta+H+\frac{1}{4m}A$, we have $\mathbf{B}(K_X+S+A+\Gamma+\frac{1}{2m}A)\subseteq\mathbf{B}(K_X+S+A+B_\delta)$, and so $S\not\subseteq\mathbf{B}(K_X+S+A+\Gamma+\frac{1}{2m}A)$. Then (24) and (25) yield

$$\begin{aligned}\mathbf{Fix}_S(K_X+S+A+\Gamma+\tfrac{1}{2m}A)&\leq\tfrac{1}{q}\operatorname{Fix}|q(K_X+S+A+B_\delta+H+\tfrac{1}{4m}A)|_S\\&\leq\mathbf{Fix}_S\big((1-\tfrac{\varepsilon}{m})(K_X+S+A+B_\delta)+G\big)\\&\leq\big(1-\tfrac{\varepsilon}{m}\big)\mathbf{F}_S(B_\delta)\leq(1-\tfrac{\varepsilon}{m})(B_{\delta|S}-C_\delta)+\mathbf{F}(C_\delta),\end{aligned}$$

and since $(1-\frac{\varepsilon}{m})\operatorname{mult}_T(B_{\delta|S}-C_\delta)\leq(1-\frac{\varepsilon}{m})\operatorname{mult}_T(B_{|S}-C)+\delta$ by assumption, to prove (22) it is enough to show that

$$\big(1-\tfrac{\varepsilon}{m}\big)\operatorname{mult}_T(B_{|S}-C)\leq\operatorname{mult}_T(\Gamma_{|S}-\Psi).$$

This is obvious if $\operatorname{mult}_T(B_{|S}-C) \leq \operatorname{mult}_T(\Gamma_{|S}-\Psi)$. Otherwise, by assumption $\phi < \operatorname{mult}_T(B_{|S}-C) \leq \operatorname{mult}_T(\Gamma_{|S}-\Psi) + \frac{\phi\varepsilon}{m}$, and so

$$\begin{aligned}(1-\tfrac{\varepsilon}{m})\operatorname{mult}_T(B_{|S}-C) &\leq \operatorname{mult}_T(\Gamma_{|S}-\Psi)+\tfrac{\phi\varepsilon}{m}-\tfrac{\varepsilon}{m}\operatorname{mult}_T(B_{|S}-C)\\ &= \operatorname{mult}_T(\Gamma_{|S}-\Psi)-\tfrac{\varepsilon}{m}\big(\operatorname{mult}_T(B_{|S}-C)-\phi\big) \leq \operatorname{mult}_T(\Gamma_{|S}-\Psi).\end{aligned}$$

Step 4. Having proved (22), we finish the proof of Claim 4.5. If T is a component of Ψ, then T is a component of C as $(\Gamma,\Psi)\in\operatorname{face}\big(\mathcal{C}_i,(B,C)\big)$. Thus $T\subseteq\operatorname{Supp}C_\delta$ for $\delta\ll 1$, and so $\operatorname{mult}_T\mathbf{F}(C_\delta)=0$ since $C_\delta\in\mathcal{F}_i$. Hence, letting $\delta\longrightarrow 0$ in (22), we get

$$\Gamma_{|S}\wedge\mathbf{Fix}_S(K_X+S+A+\Gamma+\tfrac{1}{2m}A)\leq\Gamma_{|S}-\Psi. \tag{26}$$

By Lemma 3.6, there exists a positive integer ℓ such that

$$\tfrac{1}{\ell}\operatorname{Fix}|\ell(K_X+S+A+\Gamma+\tfrac{1}{m}A)|_S\leq\mathbf{Fix}_S(K_X+S+A+\Gamma+\tfrac{1}{2m}A). \tag{27}$$

Thus, (26) and (27) give $\Psi\leq\Gamma_{|S}-\Gamma_{|S}\wedge\frac{1}{\ell}\operatorname{Fix}|\ell(K_X+S+A+\Gamma+\frac{1}{m}A)|_S$. Then Theorem 3.4 implies that $\Gamma\in\mathcal{B}^S_A(V)$ as $\Psi\in\mathcal{E}_{A_{|S}}(W)$, and furthermore, since $\mathbf{F}(\Psi)=\frac{1}{m}\operatorname{Fix}|m(K_S+A_{|S}+\Psi)|$ by assumption,

$$m\,\mathbf{F}(\Psi)+m(\Gamma_{|S}-\Psi)\geq m\,\mathbf{F}_S(\Gamma). \tag{28}$$

Since $\Psi\in\mathcal{F}_i$, we have $\Psi\wedge\mathbf{F}(\Psi)=0$, so (28) yields $\Gamma_{|S}-\Psi\geq\Gamma_{|S}\wedge\mathbf{F}_S(\Gamma)$, and finally $\Psi\leq\mathbf{\Phi}(\Gamma)$. This proves Claim 4.5.

Step 5. We now show that $\mathcal{Q}_i$ is compact and that every extreme point of $\mathcal{Q}_i$ is rational.

By abuse of notation, let $\|\cdot\|$ denote also the sup-norm on $\operatorname{Div}_{\mathbb{R}}(X)\times\operatorname{Div}_{\mathbb{R}}(S)$. Fix a point $(B,C)\in\overline{\mathcal{Q}_i}$, and let Π be the set of prime divisors T on S such that $\operatorname{mult}_T(B_{|S}-C)>0$. If $\Pi\neq\emptyset$, pick a positive rational number

$$\phi<\min\{\operatorname{mult}_T(B_{|S}-C)\mid T\in\Pi\}\leq 1,$$

and set $\phi=1$ if $\Pi=\emptyset$. By Lemma 2.14, there exist finitely many points $(\Gamma_j,\Psi_j)\in\operatorname{face}\big(\mathcal{C}_i,(B,C)\big)$ and positive integers m_j divisible by k, such that m_jA/k, $m_j\Gamma_j/k$ and $m_j\Psi_j/k$ are integral, (B,C) is a convex linear combination of all (Γ_j,Ψ_j), and

$$\|(B,C)-(\Gamma_j,\Psi_j)\|<\frac{\phi\varepsilon}{2m_j}.$$

If T is a prime divisor on S such that $T\notin\Pi$, then $\operatorname{mult}_T(\Gamma_{j|S}-\Psi_j)=0$ as $(\Gamma_j,\Psi_j)\in\operatorname{face}\big(\mathcal{C}_i,(B,C)\big)$, so Claim 4.5 implies $(\Gamma_j,\Psi_j)\in\mathcal{Q}_i'$ for all j, hence $(B,C)\in\mathcal{Q}_i$. This shows that $\mathcal{Q}_i$ is closed and that all of its extreme points are rational.

Step 6. Finally we show that $\mathcal{Q}_i$ is a rational polytope.

To this end, assume for a contradiction that $\mathcal{Q}_i$ is not a polytope. Then, by Step 5 there exist infinitely many distinct rational extreme points $v_n = (B_n, C_n)$ of $\mathcal{Q}_i$, with $n \in \mathbb{N}$. Since $\mathcal{Q}_i$ is compact and $\mathcal{C}_i$ is a rational polytope, by passing to a subsequence there exist $v_\infty = (B_\infty, C_\infty) \in \mathcal{Q}_i$ and a positive dimensional face $\mathcal{V}$ of $\mathcal{C}_i$ such that

$$v_\infty = \lim_{n\to\infty} v_n \qquad \text{and} \qquad \operatorname{face}(\mathcal{C}_i, v_n) = \mathcal{V} \quad \text{for all } n \in \mathbb{N}. \tag{29}$$

In particular, $v_\infty \in \mathcal{V}$. Let Π_∞ be the set of all prime divisors T on S such that $\operatorname{mult}_T(B_{\infty|S} - C_\infty) > 0$. If $\Pi_\infty \neq \emptyset$, pick a positive rational number

$$\phi < \min\{\operatorname{mult}_T(B_{\infty|S} - C_\infty) \mid T \in \Pi_\infty\} \leq 1,$$

and set $\phi = 1$ if $\Pi_\infty = \emptyset$. Then, by Lemma 2.14 there exist $v'_\infty \in \operatorname{face}(\mathcal{C}_i, v_\infty)$, and a positive integer m divisible by k, such that $\frac{m}{k}v'_\infty$ is integral and $\|v_\infty - v'_\infty\| < \frac{\phi\varepsilon}{2m}$. As above, by Claim 4.5 we have $v'_\infty \in \mathcal{Q}_i$. Pick $j \gg 0$ so that

$$\|v_j - v'_\infty\| \leq \|v_j - v_\infty\| + \|v_\infty - v'_\infty\| < \frac{\phi\varepsilon}{2m}, \tag{30}$$

and that $\operatorname{mult}_T(B_{j|S} - C_j) > \phi$ if $T \in \Pi_\infty$. Note that v_j is contained in the relative interior of $\mathcal{V}$ by (29), and $v'_\infty \in \operatorname{face}(\mathcal{C}_i, v_\infty) \subseteq \mathcal{V}$. Therefore, there exists a positive integer $m' \gg 0$ divisible by k, such that $\frac{m+m'}{k}v_j$ is integral, and such that if we define

$$v'_j = \frac{m+m'}{m'}v_j - \frac{m}{m'}v'_\infty \in v_j + \mathbb{R}_+(v_j - v'_\infty),$$

then $v'_j \in \mathcal{V}$. Note that $\frac{m'}{k}v'_j$ is integral,

$$v_j = \frac{m'}{m+m'}v'_j + \frac{m}{m+m'}v'_\infty, \tag{31}$$

and

$$\|v'_j - v_j\| = \frac{m}{m'}\|v_j - v'_\infty\| < \frac{\phi\varepsilon}{2m'} \tag{32}$$

by (30). Furthermore, if $v'_\infty = (B'_\infty, C'_\infty)$, $v'_j = (B'_j, C'_j)$, and if T is a prime divisor on S such that $T \notin \Pi_\infty$, then $\operatorname{mult}_T(B'_{\infty|S} - C'_\infty) = 0$ as $v'_\infty \in \operatorname{face}(\mathcal{C}_i, v_\infty)$, hence (31) gives

$$\operatorname{mult}_T(B_{j|S} - C_j) = \frac{m'}{m+m'}\operatorname{mult}_T(B'_{j|S} - C'_j) \leq \operatorname{mult}_T(B'_{j|S} - C'_j). \tag{33}$$

Therefore, $v'_j \in \mathcal{Q}_i$ by (32), (33) and by Claim 4.5, and since v_j belongs to the interior of $[v'_j, v'_\infty]$, we have that v_j is not an extreme point of $\mathcal{Q}_i$. This is a contradiction which proves the lemma. $\square$

Finally, we can proceed to the proof of Theorem 4.3.

Proof of Theorem 4.3. Step 1. In this step we prove (i). For every i, set

$$\mathcal{Q}_i' = \{(P,F) \in \mathrm{Div}_{\mathbb{Q}}(X) \times \mathrm{Div}_{\mathbb{Q}}(S) \mid P \in \mathcal{P}, F \in \mathcal{F}_i, F \leq \mathbf{\Phi}(P)\},$$

and let $\mathcal{Q}_i$ be the convex hull of $\mathcal{Q}_i'$. Then each $\mathcal{Q}_i$ is a rational polytope by Lemma 4.4.

Let $\mathcal{P}_i \subseteq V$ be the image of $\mathcal{Q}_i$ through the first projection, and denote $\mathcal{P}_{\mathbb{Q}} = \mathcal{P} \cap \mathrm{Div}_{\mathbb{Q}}(X)$. For any $P \in \mathcal{P}_{\mathbb{Q}}$ and for any sufficiently divisible positive integer m, we have $\big(P, \Phi_m(P)\big) \in \bigcup_i \mathcal{Q}_i$ by Lemma 4.2. Hence $P \in \bigcup_i \mathcal{P}_i$, and compactness implies

$$\big(P, \mathbf{\Phi}(P)\big) \in \bigcup\nolimits_i \mathcal{Q}_i. \tag{34}$$

Therefore $\mathcal{P}_{\mathbb{Q}} \subseteq \bigcup_i \mathcal{P}_i$, and since $\mathcal{P}_{\mathbb{Q}}$ is dense in $\mathcal{P}$ by Remark 2.7, we have $\mathcal{P} \subseteq \bigcup_i \mathcal{P}_i$. The reverse inclusion follows by the definition of the sets $\mathcal{Q}_i'$, and this proves (i).

Step 2. For (ii), denote $\mathcal{P}_S = S + \mathcal{P}_{\mathbb{Q}}$, and note that $\mathcal{P}_S$ lies in the hyperplane $S + V \subseteq \mathbb{R}S + V$. Fix a prime divisor $T \in W$, and consider the map $\mathbf{\Phi}_T \colon \mathcal{P}_S \longrightarrow [-1, 0]$ defined by

$$\mathbf{\Phi}_T(S+P) = -\operatorname{mult}_T \mathbf{\Phi}(P) \quad \text{for every} \quad P \in \mathcal{P}_{\mathbb{Q}}.$$

Let $\mathcal{R}_T$ be the closure of the set

$$\mathcal{R}_T' = \{S + P \in \mathcal{P}_S \mid \mathbf{\Phi}_T(S+P) \neq 0\} \subseteq \mathcal{P}_S.$$

Note that the condition $\mathbf{\Phi}_T(S+P) \neq 0$ implies $\mathbf{\Phi}_T(S+P) = -\operatorname{mult}_T \big(P_{|S} - \mathbf{F}_S(P)\big)$, and since $\mathbf{F}_S$ is a convex map on $\mathcal{P}$, the set $\mathcal{R}_T$ is convex, and $\mathbf{\Phi}_T$ is convex on $\mathcal{R}_T$.

Step 3. We first show that $\mathcal{R}_T$ is a union of some of the sets $S + \mathcal{P}_i$, and therefore that it is a rational polytope since it is convex.

To this end, fix $P \in \mathcal{P}_{\mathbb{Q}}$ such that $S + P \in \mathcal{R}_T'$. Then $\big(P, \mathbf{\Phi}(P)\big) \in \mathcal{Q}_i$ for some i by (34), and since $\operatorname{mult}_T \mathbf{\Phi}(P) \neq 0$, we have

$$\operatorname{mult}_T C > 0 \quad \text{for every point } (B, C) \text{ in the relative interior of } \mathcal{Q}_i. \tag{35}$$

Therefore, the definition of $\mathcal{F}$ yields

$$\operatorname{mult}_T \mathbf{F}(C) = 0 \quad \text{for all} \quad (B, C) \in \mathcal{Q}_i'. \tag{36}$$

Now, pick $(B, C) \in \mathcal{Q}_i'$, and let m be a positive integer such that mB/k and mC/k are integral. By Lemma 3.6, we have $\frac{1}{q} \operatorname{Fix} |q(K_X + S + A + B + \frac{1}{m}A)|_S \leq \mathbf{F}_S(B)$ for any sufficiently divisible positive integer q. Since $\mathbf{F}(C) = \frac{1}{m} \operatorname{Fix} |m(K_S + A_{|S} + C)|$ by (19), Theorem 3.4 implies

$$m\,\mathbf{F}(C) + m(B_{|S} - C) \geq m\,\mathbf{F}_S(B),$$

and hence $\operatorname{mult}_T \big(B_{|S} - \mathbf{F}_S(B)\big) \geq \operatorname{mult}_T C \geq 0$ by (36).

Therefore, for every $\mathbb{Q}$-divisor $B \in \mathcal{P}_i$ we have

$$\mathbf{\Phi}_T(S + B) = -\operatorname{mult}_T \big(B_{|S} - \mathbf{F}_S(B)\big) \leq -\operatorname{mult}_T C. \tag{37}$$

For any $\mathbb{Q}$-divisor B in the relative interior of $\mathcal{P}_i$ there exists a $\mathbb{Q}$-divisor $C \in \mathcal{F}_i$ such that (B, C) is in the relative interior of $\mathcal{Q}_i$, hence for such B we have $\mathbf{\Phi}_T(S+B) \neq 0$ by (35) and (37), that is $S + B \in \mathcal{R}'_T$. Therefore $S + \mathcal{P}_i \subseteq \mathcal{R}_T$, and $\mathcal{R}_T$ is a union of some of the sets $S + \mathcal{P}_i$.

Step 4. Next we prove that $\mathbf{\Phi}_T$ extends to a rational piecewise affine map on $\mathcal{R}_T$, and in particular that it is continuous on $\mathcal{R}_T$.

To this end, let (P_j, F_j) be the extreme points of all $\mathcal{Q}_i$ for which $S + \mathcal{P}_i \subseteq \mathcal{R}_T$. Since $\mathcal{Q}_i$ is the convex hull of $\mathcal{Q}'_i$, it follows that $(P_j, F_j) \in \bigcup \mathcal{Q}'_i$, and in particular

$$\operatorname{mult}_T F_j \leq \operatorname{mult}_T \mathbf{\Phi}(P_j) = -\mathbf{\Phi}_T(S + P_j). \tag{38}$$

Fix $P \in \mathcal{P}_\mathbb{Q}$ such that $S + P \in \mathcal{R}_T$. Then $\big(P, \mathbf{\Phi}(P)\big) \in \mathcal{Q}_i$ for some i by (34), hence there exist $r_j \in \mathbb{R}_+$ such that

$$\sum r_j = 1 \quad \text{and} \quad \big(P, \mathbf{\Phi}(P)\big) = \sum r_j (P_j, F_j).$$

Thus $\mathbf{\Phi}_T(S+P) = -\operatorname{mult}_T \mathbf{\Phi}(P) = -\sum r_j \operatorname{mult}_T F_j$, so by convexity of $\mathbf{\Phi}_T$ and by (38) we have

$$\sum r_j \mathbf{\Phi}_T(S + P_j) \geq \mathbf{\Phi}_T(S + P) = -\sum r_j \operatorname{mult}_T F_j \geq \sum r_j \mathbf{\Phi}_T(S + P_j).$$

Therefore $\mathbf{\Phi}_T(S+P_j) = -\operatorname{mult}_T F_j \in \mathbb{Q}$ for any j, and $\mathbf{\Phi}_T(S+P) = \sum r_j \mathbf{\Phi}_T(S+P_j)$. By Lemma 2.13, $\mathbf{\Phi}_T$ extends to a rational piecewise affine map on $\mathcal{R}_T$.

Step 5. Note that $\mathbf{F}_S$ is convex on $\mathcal{P}$. Thus, by definition, $\mathbf{\Phi}_T$ extends to a continuous map in the relative interior of $S + \mathcal{P}$. This, together with Step 4, implies that $\mathbf{\Phi}_T$ extends to a rational piecewise affine map on $\mathcal{P}$ for every prime divisor $T \in W$, and hence so does $\mathbf{\Phi}$, which shows the first claim in (ii).

Step 6. Finally, we show the second claim in (ii). From Step 5, in particular, we have $\mathbf{\Phi}(P) \in \operatorname{Div}_\mathbb{Q}(S)$ for every $P \in \mathcal{P}_\mathbb{Q}$, and by subdividing $\mathcal{P}$, we may assume that $\mathbf{\Phi}$ extends to a rational affine map on $\mathcal{P}$. By Lemma 2.11, the monoid $\mathbb{R}_+\mathcal{P}_S \cap \operatorname{Div}(X)$ is finitely generated, and let $q_i(S + Q_i)$ be its generators for some $q_i \in \mathbb{Q}_+$ and $Q_i \in \mathcal{P}_\mathbb{Q}$. Pick a positive integer w such that $wq_i\mathbf{\Phi}(Q_i) \in \operatorname{Div}(S)$ for every i, and set $\ell = wk$.

Fix $B \in \mathcal{P}_\mathbb{Q}$ and a positive integer m such that $\frac{m}{\ell}B \in \operatorname{Div}(X)$. If $\alpha_i \in \mathbb{N}$ are such that $\frac{m}{\ell}(S + B) = \sum \alpha_i q_i (S + Q_i)$, then $\frac{m}{\ell} = \sum \alpha_i q_i$, and therefore $\frac{m}{\ell}\mathbf{\Phi}(B) = \sum \alpha_i q_i \mathbf{\Phi}(Q_i)$ since $\mathbf{\Phi}$ is an affine map. Hence $\frac{m}{k}\mathbf{\Phi}(B) = \sum \alpha_i w q_i \mathbf{\Phi}(Q_i) \in \operatorname{Div}(S)$, so $\mathbf{F}(\mathbf{\Phi}(B)) = \frac{1}{m} \operatorname{Fix} |m(K_S + A_{|S} + \mathbf{\Phi}(B))|$ by (19). In particular,

$$\mathbf{\Phi}(B) \wedge \operatorname{Fix} |m(K_S + A_{|S} + \mathbf{\Phi}(B))| = 0 \tag{39}$$

by the definition of $\mathcal{F}$, as $\big(B, \mathbf{\Phi}(B)\big) \in \bigcup_i \mathcal{Q}_i$ by (34). By Lemma 3.6, there exists a positive integer q such that $\mathbf{\Phi}(B) \leq B_{|S} - B_{|S} \wedge \frac{1}{qm} \operatorname{Fix} |qm(K_X + S + A + B + \frac{1}{m}A)|_S$, and thus Theorem 3.4 gives

$$\begin{aligned}\operatorname{Fix}|m(K_S+A_{|S}+\mathbf{\Phi}(B))|+m(B_{|S}-\mathbf{\Phi}(B))&\geq \operatorname{Fix}|m(K_X+S+A+B)|_S\\ &\geq m(B_{|S}\wedge \tfrac{1}{m}\operatorname{Fix}|m(K_X+S+A+B)|_S)=m(B_{|S}-\Phi_m(B)).\end{aligned}$$

This together with (39) implies $\Phi_m(B)\geq \mathbf{\Phi}(B)$. But, by definition, $\mathbf{\Phi}(B)\geq \Phi_m(B)$, and (ii) follows. $\square$

Corollary 4.6. *Assume Theorem A_{n-1} and Theorem B_{n-1}.*

Let $(X,S+\sum_{i=1}^p S_i)$ be a log smooth projective pair of dimension n, where S and all S_i are distinct prime divisors. Let $V=\sum_{i=1}^p \mathbb{R}S_i\subseteq \operatorname{Div}_{\mathbb{R}}(X)$ and let A be an ample $\mathbb{Q}$-divisor on X. Then:

(i) $\mathcal{B}_A^S(V)$ is a rational polytope,

(ii) $\mathcal{B}_A^S(V)=\{B\in\mathcal{L}(V)\mid \sigma_S(K_X+S+A+B)=0\}$.

Proof. We first prove (i). Fix $B\in\overline{\mathcal{B}_A^S(V)}$, and let $B_m\in\overline{\mathcal{B}_A^S(V)}$ be a sequence of distinct points such that $\lim\limits_{m\to\infty}B_m=B$. It is enough to show that $B\in\mathcal{B}_A^S(V)$, and that for some m there exists $B_m'\in\mathcal{B}_A^S(V)$ such that $B_m\in(B,B_m')$: indeed, since B is arbitrary, this implies that $\mathcal{B}_A^S(V)$ is closed, and that around every point there are only finitely many extreme points of $\mathcal{B}_A^S(V)$. The strategy of the proof is to reduce to the situation where B is in the interior of $\mathcal{L}(V)$ and $(S,B_{|S})$ is terminal, and then to conclude by Theorem 4.3.

Let $G\in V$ be a $\mathbb{Q}$-divisor such that $B-G$ is contained in the interior of $\mathcal{L}(V)$, and that $A+G$ is ample. Denote $B^G=B-G$, $B_m^G=B_m-G$ and $A^G=A+G$, and observe that B^G and B_m^G belong to $\overline{\mathcal{B}_{A^G}^S(V)}$ for $m\gg 0$. By Lemma 2.2, there exist a log resolution $f\colon Y\longrightarrow X$ of $(X,S+B^G)$ and $\mathbb{Q}$-divisors $C,E\geq 0$ on Y with no common components, such that the components of C are disjoint, $\lfloor C\rfloor=0$, $T=f_*^{-1}S\nsubseteq\operatorname{Supp}C$, and

$$K_Y+T+C=f^*(K_X+S+B^G)+E.$$

We may then write

$$K_Y+T+C_m=f^*(K_X+S+B_m^G)+E_m,$$

where $C_m,E_m\geq 0$ are $\mathbb{Q}$-divisors on Y with no common components, $\lfloor C_m\rfloor=0$, $T\nsubseteq\operatorname{Supp}C_m$, and note that $\lim\limits_{m\to\infty}C_m=C$. Let $V^\circ\subseteq\operatorname{Div}_{\mathbb{R}}(Y)$ be the subspace spanned by the components of C and by all f-exceptional prime divisors. Then there exists an f-exceptional $\mathbb{Q}$-divisor $F\geq 0$ such that f^*A^G-F is ample, $C+F$ lies in the interior of $\mathcal{L}(V^\circ)$ and $(T,(C+F)_{|T})$ is terminal. Denote $A^\circ=f^*A^G-F$, $C^\circ=C+F$ and $C_m^\circ=C_m+F$ for all m, and observe that C° and C_m° belong to $\overline{\mathcal{B}_{A^\circ}^T(V^\circ)}$ for $m\gg 0$.

There exists a positive rational number η such that $(T,\Theta_{|T})$ is terminal for every $\Theta\in\mathcal{L}(V^\circ)$ with $\|\Theta-C^\circ\|\leq\eta$. Let $\mathcal{P}=\{\Theta\in\mathcal{L}(V^\circ)\mid \|\Theta-C^\circ\|\leq\eta\}$, and note that $\mathcal{P}$ is a rational polytope since we are working with the sup-norm. Thus $\mathcal{P}' = \mathcal{P} \cap \mathcal{B}^T_{A^\circ}(V^\circ)$ is a rational polytope by Theorem 4.3. In particular, it is closed, so C° and C°_m belong to $\mathcal{B}^T_{A^\circ}(V^\circ)$ for $m \gg 0$. Therefore, $B^G = f_*C^\circ$ and $B^G_m = f_*C^\circ_m$ belong to $\mathcal{B}^S_{A^G}(V)$ for $m \gg 0$, and hence $B, B_m \in \mathcal{B}^S_A(V)$.

Since $\mathcal{P}'$ is a polytope, by Lemma 2.8 for infinitely many m there exist $C'_m \in \mathcal{P}'$ such that $C^\circ_m \in (C^\circ, C'_m)$. Then $B^G_m \in (B^G, f_*C'_m)$, and note that $f_*C'_m \in \mathcal{B}^S_{A^G}(V)$. If we denote $B'_m = f_*C'_m + G$, then $B_m \in (B, B'_m)$ and $S \not\subseteq \mathbf{B}(K_X + S + A + B'_m)$ since $K_X + S + A + B'_m = K_X + S + A^G + f_*C'_m$. Again by Lemma 2.8 applied to the polytope $\mathcal{L}(V)$ and the point $B \in \mathcal{L}(V)$, we can assume that $B'_m \in \mathcal{L}(V)$ by choosing C'_m closer to C°. Hence $B'_m \in \mathcal{B}^S_A(V)$, and this proves (i).

Now we prove (ii). Denoting $\mathcal{Q} = \{B \in \mathcal{L}(V) \mid \sigma_S(K_X + S + A + B) = 0\}$, then clearly $\mathcal{Q} \supseteq \mathcal{B}^S_A(V)$. For the reverse inclusion, fix $B \in \mathcal{Q}$, and let H be a very ample divisor such that $(X, S + \sum_{i=1}^p S_i + H)$ is log smooth and $H \not\subseteq \operatorname{Supp}(S + \sum_{i=1}^p S_i)$. Let $V_H = \mathbb{R}H + V \subseteq \operatorname{Div}_{\mathbb{R}}(X)$, and note that $\sigma_S(K_X + S + A + B + tH) \leq \sigma_S(K_X + S + A + B) = 0$ for $t > 0$. Then $B + tH \in \mathcal{B}^S_A(V_H)$ for any $0 < t < 1$ by Lemma 2.21(i), hence $B \in \mathcal{B}^S_A(V_H)$ since $\mathcal{B}^S_A(V_H)$ is closed by the first part of the proof. Therefore $B \in \mathcal{B}^S_A(V)$. $\square$

5. Effective non-vanishing

In this section we prove that Theorem A_{n-1} and Theorem B_{n-1} imply Theorem B_n. We first sketch the idea of the proof. We consider the set

$$\mathcal{P}_A(V) = \{B \in \mathcal{L}(V) \mid K_X + A + B \equiv D \text{ for some } \mathbb{R}\text{-divisor } D \geq 0\},$$

and prove that it is a rational polytope. Once we know that $\mathcal{P}_A(V)$ is a rational polytope, it is a straightforward application of the Kawamata-Viehweg vanishing to show that this set coincides with $\mathcal{E}_A(V)$, see Lemma 5.1.

In order to show that $\mathcal{P}_A(V)$ is a rational polytope, we first show that if an adjoint divisor $K_X + A + B$ is pseudo-effective, then it is numerically equivalent to an effective divisor, which in particular implies that the set $\mathcal{P}_A(V)$ is compact. This statement is usually referred to as "non-vanishing." We may assume that $K_X + A + B \not\equiv N_\sigma(K_X + A + B)$, and the claim is a consequence of Corollary 4.6, see Lemma 5.3.

Then, we show that $\mathcal{P}_A(V)$ is a polytope (rationality of this polytope is easy): we assume for contradiction that there are infinitely many exteme points B_m of $\mathcal{P}_A(V)$, and by compactness and by passing to a subsequence we can assume that they converge to a point $B \in \mathcal{P}_A(V)$. We can then derive a contradiction if we can show that for some $m \gg 0$ there is a point $B'_m \in \mathcal{P}_A(V)$ such that $B_m \in (B, B'_m)$.

This is straightforward when $K_X + A + B \equiv N_\sigma(K_X + A + B)$, and the difficult case is when $K_X + A + B \not\equiv N_\sigma(K_X + A + B)$. We consider the cones

$$\mathcal{C} = \mathbb{R}_+(K_X + A + \mathcal{P}_A(V)) \quad \text{and} \quad \mathcal{C}_S = \mathbb{R}_+(K_X + S + A + \mathcal{B}^S_A(V)) \subseteq \mathcal{C}$$

for some prime divisor S, and note that $\mathcal{C}_S$ is a rational polyhedral cone by Corollary 4.6. We proceed in two steps.

First, non-vanishing implies that there exists an $\mathbb{R}$-divisor $F \geq 0$ such that $K_X + A + B \sim_{\mathbb{R}} F$. We then find a divisor $\Lambda \geq 0$, whose support is contained in the support of $N_\sigma(K_X + A + B)$, and a positive real number μ such that

$$\Sigma = (1+\mu)(K_X + A + B) - \Lambda = (1+\mu)(K_X + A + B - \tfrac{1}{1+\mu}\Lambda) \in \mathcal{C}_S$$

for some prime divisor S contained in the support of F. Then it is easy to find rational numbers ε_m which converge to 1 such that the divisors

$$\Sigma_m = (1+\mu)(K_X + A + B_m) - \varepsilon_m\Lambda = (1+\mu)(K_X + A + B_m - \tfrac{\varepsilon_m}{1+\mu}\Lambda)$$

are pseudo-effective. Much of the proof of Theorem 5.5 is devoted to proving these facts. Note that even though the divisor $B - \frac{1}{1+\mu}\Lambda$ belongs to $\mathcal{L}(V)$, that is not necessarily the case with divisors $B_m - \frac{\varepsilon_m}{1+\mu}\Lambda$.

Second, in order to show that there is a point $B_m \in (B, B'_m)$ as above, it suffices to find a pseudo-effective divisor Σ'_m for some $m \gg 0$ such that $\Sigma_m \in (\Sigma, \Sigma'_m)$, and this is done in Lemma 5.4, using the fact that $\mathcal{C}_S$ is a rational polyhedral cone.

We start with the following lemma which uses ideas from Shokurov's proof of the classical non-vanishing theorem.

Lemma 5.1. *Let (X, B) be a log smooth pair, where B is a $\mathbb{Q}$-divisor such that $\lfloor B \rfloor = 0$. Let A be a nef and big $\mathbb{Q}$-divisor, and assume that there exists an $\mathbb{R}$-divisor $D \geq 0$ such that $K_X + A + B \equiv D$.*

Then there exists a $\mathbb{Q}$-divisor $D' \geq 0$ such that $K_X + A + B \sim_{\mathbb{Q}} D'$.

Proof. Let $V \subseteq \operatorname{Div}(X)_{\mathbb{R}}$ be the vector space spanned by the components of K_X, A, B and D, and let $\phi\colon V \longrightarrow N^1(X)_{\mathbb{R}}$ be the linear map sending an $\mathbb{R}$-divisor to its numerical class. Since $\phi^{-1}(\phi(K_X + A + B))$ is a rational affine subspace of V, we can assume that $D \geq 0$ is a $\mathbb{Q}$-divisor.

First assume that $(X, B + D)$ is log smooth. Let m be a positive integer such that $m(A + B)$ and mD are integral. Denoting $F = (m-1)D + B$, $L = m(K_X + A + B) - \lfloor F \rfloor$ and $L' = mD - \lfloor F \rfloor$, we have

$$L \equiv L' = D - B + \{F\} \equiv K_X + A + \{F\}.$$

Thus, Kawamata-Viehweg vanishing implies that $h^i(X, \mathcal{O}_X(L)) = h^i(X, \mathcal{O}_X(L')) = 0$ for all $i > 0$, and since the Euler characteristic is a numerical invariant, this yields $h^0(X, \mathcal{O}_X(L)) = h^0(X, \mathcal{O}_X(L'))$. As mD is integral and $\lfloor B \rfloor = 0$, it follows that

$$L' = mD - \lfloor (m-1)D + B \rfloor = \lceil D - B \rceil \geq 0,$$

and thus $h^0\big(X, \mathcal{O}_X(m(K_X + A + B))\big) = h^0(X, \mathcal{O}_X(L + \lfloor F \rfloor)) \geq h^0(X, \mathcal{O}_X(L)) = h^0(X, \mathcal{O}_X(L')) > 0$.

In the general case, let $f\colon Y \longrightarrow X$ be a log resolution of $(X, B + D)$. Then there exist $\mathbb{Q}$-divisors $B', E \geq 0$ with no common components such that E is f-exceptional and $K_Y + B' = f^*(K_X + B) + E$. Therefore $K_Y + f^*A + B' \equiv f^*D + E \geq 0$, so by above there exists a $\mathbb{Q}$-divisor $D^\circ \geq 0$ such that $K_Y + f^*A + B' \sim_{\mathbb{Q}} D^\circ$. Hence $K_X + A + B \sim_{\mathbb{Q}} f_*D^\circ \geq 0$. □

Lemma 5.2. *Let X be a smooth projective variety of dimension n and let $x \in X$. Let $D \in \operatorname{Div}(X)$ and assume that s is a positive integer such that $h^0(X, \mathcal{O}_X(D)) > \binom{s+n}{n}$.*

Then there exists $D' \in |D|$ such that $\operatorname{mult}_x D' > s$.

Proof. Let $\mathfrak{m} \subseteq \mathcal{O}_X$ be the ideal sheaf of x. Then we have

$$h^0(X, \mathcal{O}_X/\mathfrak{m}^{s+1}) = \dim_{\mathbb{C}} \mathbb{C}[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^{s+1} = \binom{s+n}{n},$$

hence $h^0(X, \mathcal{O}_X(D)) > h^0(X, \mathcal{O}_X/\mathfrak{m}^{s+1})$. Therefore the exact sequence

$$0 \longrightarrow \mathfrak{m}^{s+1} \otimes \mathcal{O}_X(D) \longrightarrow \mathcal{O}_X(D) \longrightarrow (\mathcal{O}_X/\mathfrak{m}^{s+1}) \otimes \mathcal{O}_X(D) \simeq \mathcal{O}_X/\mathfrak{m}^{s+1} \longrightarrow 0$$

yields $h^0\big(X, \mathfrak{m}^{s+1} \otimes \mathcal{O}_X(D)\big) > 0$, so there exists a divisor $D' \in |D|$ with multiplicity at least $s+1$ at x. □

Lemma 5.3. *Assume Theorem A_{n-1} and Theorem B_{n-1}.*

Let (X, B) be a log smooth pair of dimension n, where B is an $\mathbb{R}$-divisor such that $\lfloor B \rfloor = 0$. Let A be an ample $\mathbb{Q}$-divisor on X, and assume that $K_X + A + B$ is a pseudo-effective $\mathbb{R}$-divisor such that $K_X + A + B \not\equiv N_\sigma(K_X + A + B)$.

Then there exists an $\mathbb{R}$-divisor $F \geq 0$ such that $K_X + A + B \sim_{\mathbb{R}} F$.

Proof. By Lemma 2.19, we have $h^0(X, \mathcal{O}_X(\lfloor mk(K_X + A + B) \rfloor + kA)) > \binom{nk+n}{n}$ for any sufficiently divisible positive integers m and k. Fix a point $x \in X \setminus \bigcup_{\varepsilon > 0} \mathbf{B}(K_X + A + B + \varepsilon A)$. Then, by Lemma 5.2 there exists an $\mathbb{R}$-divisor $G \geq 0$ such that $G \sim_{\mathbb{R}} mk(K_X + A + B) + kA$ and $\operatorname{mult}_x G > nk$, so setting $D = \frac{1}{mk} G$, we have

$$D \sim_{\mathbb{R}} K_X + A + B + \frac{1}{m} A \qquad \text{and} \qquad \operatorname{mult}_x D > \frac{n}{m}. \tag{40}$$

For any $t \in [0, m]$, define $A_t = \frac{m-t}{m} A$ and $\Psi_t = B + tD$, so that

$$(1+t)(K_X + A + B) \sim_{\mathbb{R}} K_X + A + B + t\big(D - \tfrac{1}{m} A\big) = K_X + A_t + \Psi_t. \tag{41}$$

Let $f \colon Y \longrightarrow X$ be a log resolution of $(X, B + D)$ constructed by first blowing up X at x. Then for every $t \in [0, m]$, there exist $\mathbb{R}$-divisors $C_t, E_t \geq 0$ with no common components such that E_t is f-exceptional and

$$K_Y + C_t = f^*(K_X + \Psi_t) + E_t. \tag{42}$$

The exceptional divisor of the initial blowup gives a prime divisor $P \subseteq Y$ such that $\operatorname{mult}_P(K_Y - f^*K_X) = n - 1$, $\operatorname{mult}_P f^*\Psi_t = \operatorname{mult}_x \Psi_t$, and $P \notin \operatorname{Supp} N_\sigma(f^*(K_X + A + B))$ by Remark 2.18. Since $\operatorname{mult}_x \Psi_m > n$ by (40), it follows from (42) that

$$\operatorname{mult}_P E_m = 0 \quad \text{and} \quad \operatorname{mult}_P C_m > 1. \tag{43}$$

Note that $\lfloor C_0 \rfloor = 0$, and denote

$$B_t = C_t - C_t \wedge N_\sigma(K_Y + f^*A_t + C_t).$$

Observe that by (41) and (42) we have

$$\begin{aligned} N_\sigma(K_Y + f^*A_t + C_t) &= N_\sigma\big(f^*(K_X + A_t + \Psi_t)\big) + E_t \\ &= (1+t)N_\sigma\big(f^*(K_X + A + B)\big) + E_t, \end{aligned}$$

hence B_t is a continuous function in t. Moreover $P \not\subseteq \operatorname{Supp} N_\sigma(K_Y + f^*A_m + B_m)$ by the choice of x and by (43), and in particular $\operatorname{mult}_P B_m > 1$. Pick $0 < \varepsilon \ll 1$ such that $\operatorname{mult}_P B_{m-\varepsilon} > 1$, and let $H \geq 0$ be an f-exceptional $\mathbb{Q}$-divisor on Y such that $\lfloor B_0 + H \rfloor = 0$ and $f^*A_{m-\varepsilon} - H$ is ample. Then there exists a minimal $\lambda < m - \varepsilon$ such that $\lfloor B_\lambda + H \rfloor \neq 0$, and let $S \subseteq \lfloor B_\lambda + H \rfloor$ be a prime divisor. Since $\lfloor H \rfloor = 0$, we have $S \subseteq \operatorname{Supp} B_\lambda$. As $B_\lambda \wedge N_\sigma(K_Y + f^*A_\lambda + B_\lambda) = 0$ by Lemma 2.16, we deduce that $S \not\subseteq \operatorname{Supp} N_\sigma(K_Y + f^*A_\lambda + B_\lambda)$.

Let $A' = f^*A_\lambda - H = f^*(\frac{m-\varepsilon-\lambda}{m}A) + (f^*A_{m-\varepsilon} - H)$. Then A' is ample, and since $\sigma_S(K_Y + A' + B_\lambda + H) = \sigma_S(K_Y + f^*A_\lambda + B_\lambda) = 0$ by what we proved above, Corollary 4.6 implies that $S \not\subseteq \mathbf{B}(K_Y + A' + B_\lambda + H) = \mathbf{B}(K_Y + f^*A_\lambda + B_\lambda)$. In particular, there exists an $\mathbb{R}$-divisor $F' \geq 0$ such that $K_Y + f^*A_\lambda + B_\lambda \sim_\mathbb{R} F'$, and thus, by (41) and (42),

$$K_X + \Delta \sim_\mathbb{R} \frac{1}{1+\lambda} f_*(K_Y + f^*A_\lambda + C_\lambda) \sim_\mathbb{R} \frac{1}{1+\lambda} f_*(F' + C_\lambda - B_\lambda) \geq 0.$$

This finishes the proof. $\square$

Lemma 5.4. *Assume Theorem A_{n-1} and Theorem B_{n-1}.*

Let $(X, S + \sum_{i=1}^p S_i)$ be a log smooth projective pair of dimension n, where S and the S_i are distinct prime divisors. Let A be an ample $\mathbb{Q}$-divisor on X, let $W = \mathbb{R}S + \sum_{i=1}^p \mathbb{R}S_i \subseteq \operatorname{Div}_\mathbb{R}(X)$, and assume $\Upsilon \in \mathcal{L}(W)$ and $0 \leq \Sigma \in W$ are such that

$$\operatorname{mult}_S \Upsilon = 1,\ \operatorname{mult}_S \Sigma > 0,\ \sigma_S(K_X + A + \Upsilon) = 0 \quad \textit{and} \quad K_X + A + \Upsilon \sim_\mathbb{R} \Sigma.$$

Let $\Upsilon_m \in W$ be a sequence such that $K_X + A + \Upsilon_m$ are pseudo-effective and $\lim\limits_{m\to\infty} \Upsilon_m = \Upsilon$.

Then for infinitely many m there exist $\Upsilon'_m \in W$ such that $\Upsilon_m \in (\Upsilon, \Upsilon'_m)$ and $K_X + A + \Upsilon'_m$ are pseudo-effective.

Proof. Step 1. Denote

$$\Sigma_m = \Sigma + \Upsilon_m - \Upsilon.$$

Then $\Sigma_m \sim_\mathbb{R} K_X + A + \Upsilon_m$ is pseudo-effective by assumption, and hence so is

$$\Gamma_m = \Sigma_m - \sigma_S(\Sigma_m) \cdot S.$$

In Step 2, we will construct a rational polytope $\mathcal{P}$ which does not contain the origin, and the rational polyhedral cone $\mathcal{D} = \mathbb{R}_+\mathcal{P} \subseteq W$ such that every element of $\mathcal{D}$ is pseudo-effective and, after passing to a subsequence,

$$\Gamma_m \in \mathcal{D} \quad \text{for all } m > 0, \text{ and } \quad \lim_{m\to\infty} \Gamma_m = \Sigma. \tag{44}$$

This immediately implies the lemma: indeed, Remark 2.9 applied to $\mathcal{D}$ and to the point $\Sigma \in \mathcal{D}$ shows that for any $m \gg 0$ there exist $\Psi_m \in \mathcal{D}$ and $0 < \mu_m < 1$ such that $\Gamma_m = \mu_m\Sigma + (1-\mu_m)\Psi_m$. Then Ψ_m is pseudo-effective, and thus so is the $\mathbb{R}$-divisor

$$\Sigma'_m = \Psi_m + \frac{1}{1-\mu_m}(\Sigma_m - \Gamma_m) = \Psi_m + \frac{\sigma_S(\Sigma_m)}{1-\mu_m}S.$$

Let $\Upsilon'_m = \frac{1}{1-\mu_m}(\Upsilon_m - \mu_m\Upsilon) \in W$. Then it is easy to check that $\Upsilon_m \in (\Upsilon, \Upsilon'_m)$ and $K_X + A + \Upsilon'_m \sim_{\mathbb{R}} \Sigma'_m$, and we are done.

Step 2. In this step, we construct a rational polytope $\mathcal{D}$ with required properties. Denote $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \mathrm{Div}_{\mathbb{R}}(X)$. Let $Z = \sum\limits_{\mathrm{mult}_{S_j}\Upsilon=0} S_j - \sum\limits_{\mathrm{mult}_{S_i}\Upsilon=1} S_i$, and pick a rational number $0 < \varepsilon \ll 1$ such that the $\mathbb{Q}$-divisor $A' = A - \varepsilon Z$ is ample. Setting $\Upsilon' = \Upsilon - S + \varepsilon Z$, we have

$$\Upsilon' \in \sum_{i=1}^p [\varepsilon S_i, (1-\varepsilon)S_i] \tag{45}$$

and

$$K_X + S + A' + \Upsilon' \sim_{\mathbb{R}} \Sigma. \tag{46}$$

By Corollary 4.6, $\mathcal{B}^S_{A'}(V)$ is a rational polytope, and denote

$$\mathcal{P} = \Sigma - \Upsilon' + \mathcal{B}^S_{A'}(V) \quad \text{and} \quad \mathcal{D} = \mathbb{R}_+\mathcal{P} \subseteq W.$$

Then $\mathcal{P}$ is a rational polytope and $\mathcal{D}$ is a rational polyhedral cone. Since $\sigma_S(K_X + S + A' + \Upsilon') = \sigma_S(\Sigma) = \sigma_S(K_X + A + \Upsilon) = 0$ by assumption, Corollary 4.6 implies that $\Upsilon' \in \mathcal{B}^S_{A'}(V)$, and therefore $\Sigma \in \mathcal{P}$. By the definition of $\mathcal{P}$ and by (46), for every $D \in \mathcal{P}$ there exists $B \in \mathcal{B}^S_{A'}(V)$ such that

$$D = \Sigma - \Upsilon' + B \sim_{\mathbb{R}} K_X + S + A' + B.$$

Since $\mathrm{mult}_S \Upsilon' = \mathrm{mult}_S B = 0$, this implies $\mathrm{mult}_S D = \mathrm{mult}_S \Sigma > 0$ and, in particular, $\mathcal{P}$ does not contain the origin. Moreover, by the definition of $\mathcal{B}^S_{A'}(V)$, every D is pseudo-effective, hence every element of $\mathcal{D}$ is pseudo-effective.

Now we prove (44). Let $\lambda_m = \mathrm{mult}_S \Gamma_m / \mathrm{mult}_S \Sigma \in \mathbb{R}$, and for every m choose $0 < \beta_m \ll 1$ such that $\beta_m\lambda_m < 1$ and $\beta_m\|\Gamma_m - \lambda_m\Sigma\| < \varepsilon$. Set $\delta_m = \beta_m\lambda_m$ and $t_m = (1-\delta_m)/(1-\delta_m+\beta_m)$, and note that $0 < t_m < 1$. We first show that

$$t_m\Sigma + (1-t_m)\Gamma_m \in \mathcal{D}. \tag{47}$$

To this end, denote $R_m = \Upsilon' + \beta_m\Gamma_m - \delta_m\Sigma$, and note that by the choice of β_m and δ_m we have $\operatorname{mult}_S R_m = 0$. Furthermore, since $\|\beta_m\Gamma_m - \delta_m\Sigma\| < \varepsilon$, by (45) we have $R_m \in \mathcal{L}(V)$. Then (46) implies

$$t_m\Sigma + (1-t_m)\Gamma_m = \frac{1}{1-\delta_m+\beta_m}(\Sigma - \Upsilon' + R_m) \tag{48}$$

$$\sim_{\mathbb{R}} \frac{1}{1-\delta_m+\beta_m}(K_X + S + A' + R_m), \tag{49}$$

and observe that by assumption and by definition of Γ_m, we have

$$\sigma_S\big(t_m\Sigma + (1-t_m)\Gamma_m\big) \leq t_m\sigma_S(\Sigma) + (1-t_m)\sigma_S(\Gamma_m) = 0. \tag{50}$$

Therefore $R_m \in \mathcal{B}^S_{A'}(V)$ by Corollary 4.6, by (49), and by (50), hence (48) and the definition of $\mathcal{D}$ imply (47).

In particular, $(\Sigma, \Gamma_m) \cap \mathcal{D} \neq \emptyset$. By Remark 2.17, the sequence $\sigma_S(\Sigma_m)$ is bounded. Therefore, by Lemma 2.10 and after passing to a subsequence we may assume that there exists $P_m \in [\Sigma_m, \Gamma_m] \cap \mathcal{D}$ for every m. By the definition of $\mathcal{D}$, there exists $r_m \in \mathbb{R}_+$ and $B_m \in \mathcal{B}^S_{A'}(V)$ such that $P_m = r_m(K_X+S+A'+B_m)$, hence $\sigma_S(P_m) = 0$ by Corollary 4.6. Thus, the definition of Γ_m implies $P_m = \Gamma_m$, and $(\Sigma_m, \Gamma_m) \cap \mathcal{D} = \emptyset$. Then (44) follows from Lemma 2.10 again applied to $\mathcal{D}$, and the proof is complete. □

Theorem 5.5. *Theorem A_{n-1} and Theorem B_{n-1} imply Theorem B_n.*

Proof. Let

$$\mathcal{P}_A(V) = \{B \in \mathcal{L}(V) \mid K_X + A + B \equiv D \text{ for some } \mathbb{R}\text{-divisor } D \geq 0\}.$$

We claim that $\mathcal{P}_A(V)$ is a rational polytope. Assuming this, let $B_1, \dots, B_q$ be the extreme points of $\mathcal{P}_A(V)$, and choose $\varepsilon > 0$ such that $A + \varepsilon B_i$ is ample for every i. Since $K_X + A + B_i = K_X + (A + \varepsilon B_i) + (1-\varepsilon)B_i$ and $\lfloor (1-\varepsilon)B_i \rfloor = 0$, Lemma 5.1 implies that there exist $\mathbb{Q}$-divisors $D_i \geq 0$ such that $K_X + A + B_i \sim_{\mathbb{Q}} D_i$. Thus $B_i \in \mathcal{E}_A(V)$ for every i, and therefore $\mathcal{P}_A(V) \subseteq \mathcal{E}_A(V)$ as $\mathcal{E}_A(V)$ is convex. Since obviously $\mathcal{E}_A(V) \subseteq \mathcal{P}_A(V)$, the theorem follows.

Now we prove that $\mathcal{P}_A(V)$ is a rational polytope in several steps.

Step 1. In this step we show that $\mathcal{P}_A(V)$ is closed. To this end, fix $B \in \overline{\mathcal{P}_A(V)}$ and denote $\Delta = A + B$. In particular, $K_X + \Delta$ is pseudo-effective. If $K_X + \Delta \equiv N_\sigma(K_X+\Delta)$, then it follows immediately that $B \in \mathcal{P}_A(V)$. If $K_X+\Delta \not\equiv N_\sigma(K_X+\Delta)$, assume first that $\lfloor B \rfloor = 0$. Then by Lemma 5.3 there exists an $\mathbb{R}$-divisor $F \geq 0$ such that $K_X + \Delta \sim_{\mathbb{R}} F$, and in particular $B \in \mathcal{P}_A(V)$. If $\lfloor B \rfloor \neq 0$, pick a $\mathbb{Q}$-divisor $0 \leq G \in V$ such that $A + G$ is ample and $\lfloor B - G \rfloor = 0$. Then $B - G \in \mathcal{P}_{A+G}(V)$ by above, and hence $B \in \mathcal{P}_A(V)$. This implies that $\mathcal{P}_A(V)$ is compact.

Step 2. We next show that $\mathcal{P}_A(V)$ is a polytope. Assume for contradiction that $\mathcal{P}_A(V)$ is not a polytope. Then there exists an infinite sequence of distinct extreme points $B_m \in \mathcal{P}_A(V)$. By compactness and by passing to a subsequence we can assume that there is a point $B \in \mathcal{P}_A(V)$ such that $\lim_{m\to\infty} B_m = B$. Therefore, in order to derive a contradiction, it is enough to prove the following.

Claim 5.6. *Fix $B \in \mathcal{P}_A(V)$, and let $B_m \in \mathcal{P}_A(V)$ be a sequence of distinct points such that $\lim_{m\to\infty} B_m = B$. Then for infinitely many m there exist $B'_m \in \mathcal{P}_A(V)$ such that $B_m \in (B, B'_m)$.*

We remark that it is enough to find *one* such m, however the use of Lemma 2.8 in Step 5 shows that we need this stronger version of the claim.

We prove the claim in the following three steps. In Steps 3 and 4 we assume that $\lfloor B\rfloor = 0$, and in Step 5 we reduce the general case to this one.

Step 3. In this step we assume that $\lfloor B\rfloor = 0$ and

$$K_X + A + B \not\equiv N_\sigma(K_X + A + B). \tag{51}$$

By Lemma 5.3, there exists an $\mathbb{R}$-divisor $F \geq 0$ such that

$$K_X + A + B \sim_{\mathbb{R}} F. \tag{52}$$

We first prove Claim 5.6 under an additional assumption that $F \in V$, and treat the general case at the end of Step 3.

For any $t \geq 0$, define

$$\Phi_t = B + tF, \tag{53}$$

so that by (52),

$$(1+t)(K_X + A + B) \sim_{\mathbb{R}} K_X + A + B + tF = K_X + A + \Phi_t. \tag{54}$$

Note that $\lfloor \Phi_0 \rfloor = 0$ and

$$N_\sigma(K_X + A + \Phi_t) = (1+t)N_\sigma(K_X + A + B) = (1+t)N_\sigma(F). \tag{55}$$

Thus, if we denote

$$\Upsilon_t = \Phi_t - \Phi_t \wedge N_\sigma(K_X + A + \Phi_t), \tag{56}$$

then Υ_t is a continuous function in t. Write $F = \sum_{j=1}^{\ell} f_j F_j$, where F_j are prime divisors and $f_j > 0$ for all j. Since $F \not\equiv N_\sigma(F)$ by (51) and (52), Lemma 2.20 implies that there exists $j \in \{1, \dots, \ell\}$ such that $\sigma_{F_j}(F) = 0$. Thus, by (53), (55) and (56),

$$\operatorname{mult}_{F_j} \Upsilon_t = \operatorname{mult}_{F_j} B + t f_j,$$

so there exists a minimal $\mu > 0$ such that $\lfloor \Upsilon_\mu \rfloor \neq 0$. Note that $\lfloor \Upsilon_\mu \rfloor \subseteq \operatorname{Supp} F$, but F_j is not necessarily a component of $\lfloor \Upsilon_\mu \rfloor$. Let $S \subseteq \lfloor \Upsilon_\mu \rfloor$ be a prime divisor. Observe that

$$\sigma_S(K_X + A + \Upsilon_\mu) = 0 \tag{57}$$

by (56), and

$$\sigma_S\big((1+\mu)F\big) = \sigma_S(K_X+A+\Phi_\mu) = \operatorname{mult}_S\Phi_\mu - \operatorname{mult}_S\Upsilon_\mu \\ = \operatorname{mult}_S B + \mu\operatorname{mult}_S F - 1 < \mu\operatorname{mult}_S F \tag{58}$$

by (53), (55) and (56). Let $\Sigma = (1+\mu)F - \Phi_\mu \wedge N_\sigma\big((1+\mu)F\big)$. Then we have

$$\Sigma \geq 0 \quad \text{and} \quad K_X + A + \Upsilon_\mu \sim_{\mathbb{R}} \Sigma \tag{59}$$

by (55) and (56), and moreover,

$$\operatorname{mult}_S\Sigma \geq (1+\mu)\operatorname{mult}_S F - \sigma_S\big((1+\mu)F\big) > \operatorname{mult}_S F \geq 0 \tag{60}$$

by (58). For every $m\in\mathbb{N}$, define $\Phi_{\mu,m} = B_m + \mu(F + B_m - B)$. Then

$$\lim_{m\to\infty}\Phi_{\mu,m} = \Phi_\mu \qquad \text{and} \qquad (1+\mu)(K_X+A+B_m) \sim_{\mathbb{R}} K_X + A + \Phi_{\mu,m} \tag{61}$$

by assumption, and by (53) and (52). Let

$$\Lambda = \Phi_\mu \wedge N_\sigma(K_X+A+\Phi_\mu) \quad \text{and} \quad \Lambda_m = \Phi_{\mu,m} \wedge \sum_{Z\subseteq \operatorname{Supp}\Lambda} \sigma_Z(K_X+A+\Phi_{\mu,m})\cdot Z.$$

Note that $0 \leq \Lambda_m \leq N_\sigma(K_X+A+\Phi_{\mu,m})$, and therefore $K_X+A+\Phi_{\mu,m}-\Lambda_m$ is pseudo-effective by Lemma 2.16. By Lemma 2.16 again, we have $\Lambda \leq \liminf\limits_{m\to\infty}\Lambda_m$, and in particular, $\operatorname{Supp}\Lambda_m = \operatorname{Supp}\Lambda$ for $m \gg 0$. Thus, there exists an increasing sequence of rational numbers $\varepsilon_m > 0$ such that $\lim\limits_{m\to\infty}\varepsilon_m = 1$ and $\Lambda_m \geq \varepsilon_m\Lambda$.

Define $\Upsilon_{\mu,m} = \Phi_{\mu,m} - \varepsilon_m\Lambda$. Note that

$$K_X + A + \Upsilon_{\mu,m} \quad \text{is pseudo-effective} \qquad \text{and} \qquad \lim_{m\to\infty}\Upsilon_{\mu,m} = \Phi_\mu - \Lambda = \Upsilon_\mu \tag{62}$$

by (61) and (56). Therefore, by (57), (59), (60), (62) and Lemma 5.4, and by passing to a subsequence, for every m there exist $\Upsilon'_m \in V$ and $0 < \alpha_m \ll 1$ such that

$$K_X + A + \Upsilon'_m \quad \text{is pseudo-effective} \qquad \text{and} \qquad \Upsilon_{\mu,m} = \alpha_m\Upsilon_\mu + (1-\alpha_m)\Upsilon'_m.$$

Setting $B'_m = \frac{1}{1-\alpha_m}(B_m - \alpha_m B)$, we have $B_m = \alpha_m B + (1-\alpha_m)B'_m$, and an easy calculation involving (53), (61) and (62) shows that

$$K_X + A + B'_m \sim_{\mathbb{R}} \frac{1}{1+\mu}\Big(K_X + A + \Upsilon'_m + \frac{\varepsilon_m - \alpha_m}{1-\alpha_m}\Lambda\Big).$$

In particular, $K_X + A + B'_m$ is pseudo-effective for $m \gg 0$. Since $\mathcal{L}(V)$ is a rational polytope, Lemma 2.8 yields $B'_m \in \mathcal{L}(V)$ for $m \gg 0$, which proves Claim 5.6 under the additional assumption that $F \in V$.

To show the general case, let $f\colon Y \longrightarrow X$ be a log resolution of $(X, B+F)$. Then there are $\mathbb{R}$-divisors $C, E \geq 0$ on Y with no common components and $C_m, E_m \geq 0$ on Y with no common components such that E and E_m are f-exceptional and

$$K_Y + C = f^*(K_X+B) + E \quad \text{and} \quad K_Y + C_m = f^*(K_X+B_m) + E_m.$$

Note that $\lim_{m\to\infty} C_m = C$. Let $G \geq 0$ be an f-exceptional $\mathbb{Q}$-divisor on Y such that A° is ample, $\lfloor C^\circ \rfloor = 0$, and $\lfloor C_m^\circ \rfloor = 0$ for all $m \gg 0$, where $A^\circ = f^*A - G$, $C^\circ = C + G$ and $C_m^\circ = C_m + G$. Denoting $F^\circ = f^*F + E \geq 0$, we have

$$f_*C^\circ = B, \quad f_*C_m^\circ = B_m, \quad \text{and} \quad K_Y + A^\circ + C^\circ \sim_{\mathbb{R}} F^\circ.$$

Let $V^\circ \subseteq \mathrm{Div}_{\mathbb{R}}(Y)$ be the vector space spanned by the components of $\sum_{i=1}^p f_*^{-1}S_i + f_*^{-1}F$ plus all exceptional prime divisors, and note that $F^\circ \in V^\circ$. By what we proved above, for infinitely many m there exist $C'_m \in \mathcal{P}_{A^\circ}(V^\circ)$ such that $C_m^\circ \in (C^\circ, C'_m)$. Note that $\operatorname{Supp} C'_m$ is a subset of $\sum_{i=1}^p f_*^{-1}S_i$ plus all exceptional prime divisors, and denote $B'_m = f_*C'_m \in \mathcal{L}(V)$. Then $B_m \in (B, B'_m)$, and $K_X + A + B'_m = f_*(K_Y + A^\circ + C'_m)$ is numerically equivalent to an effective divisor, hence $B'_m \in \mathcal{P}_A(V)$, finishing the proof of Claim 5.6 when $\lfloor B \rfloor = 0$ and $K_X + A + B \not\equiv N_\sigma(K_X + A + B)$.

Step 4. Now assume that $\lfloor B \rfloor = 0$ and $K_X + A + B \equiv N_\sigma(K_X + A + B)$. Let $D_m \geq 0$ be $\mathbb{R}$-divisors such that $K_X + A + B_m \equiv D_m$. By Lemma 2.21(ii), there exists an ample $\mathbb{R}$-divisor H such that

$$\operatorname{Supp} N_\sigma(K_X + A + B) \subseteq \mathbf{B}(K_X + A + B + H),$$

and as $H + (K_X + A + B - D_m) \equiv H + (B - B_m)$ is ample for all $m \gg 0$, by passing to a subsequence we may assume that

$$\begin{aligned} \operatorname{Supp} N_\sigma(K_X + A + B) &\subseteq \mathbf{B}\big(D_m + H + (K_X + A + B - D_m)\big) \\ &\subseteq \mathbf{B}(D_m) \subseteq \operatorname{Supp} D_m \end{aligned} \tag{63}$$

for all m. For $m \in \mathbb{N}$ and $t > 1$, denote $C_{m,t} = B + t(B_m - B)$, and observe that

$$B_m = \frac{1}{t}C_{m,t} + \frac{t-1}{t}B \tag{64}$$

and

$$K_X + A + C_{m,t} \equiv tD_m - (t-1)(K_X + A + B) \equiv tD_m - (t-1)N_\sigma(K_X + A + B). \tag{65}$$

Since $\mathcal{L}(V)$ is a polytope and $B \in \mathcal{L}(V)$, pick $\delta = \delta(B, \mathcal{L}(V)) > 0$ as in Lemma 2.8. By passing to a subsequence we may assume that $\|B_m - B\| \leq \delta/2$ for every m, and as $\|C_{m,t} - B\| = t\|B_m - B\|$, Lemma 2.8 gives $C_{m,t} \in \mathcal{L}(V)$ for all m and $1 < t < 2$.

Fix m. By (63) there exists $1 < t_m < 2$ such that $t_mD_m - (t_m - 1)N_\sigma(K_X + A + B) \geq 0$, and denote $B'_m = C_{m,t_m}$. Then (65) implies $B'_m \in \mathcal{P}_A(V)$, and thus (64) proves Claim 5.6.

Step 5. Now we treat the general case of Claim 5.6. Pick $\delta = \delta(B, \mathcal{L}(V))$ as in Lemma 2.8. By passing to a subsequence, we may choose a $\mathbb{Q}$-divisor $0 \leq G \in V$ such that A° is ample, $\lfloor B^\circ \rfloor = 0$ and all $\lfloor B_m^\circ \rfloor = 0$, where $A^\circ = A + G$, $B^\circ = B - G$ and $B_m^\circ = B_m - G$. By Steps 3 and 4, for infinitely many m there exist $F_m \in \mathcal{P}_{A^\circ}(V)$ such that $B_m^\circ \in (B^\circ, F_m)$. In particular, setting $B'_m = F_m + G$, we have $B_m \in (B, B'_m)$. Since $B - B'_m = B^\circ - F_m$, we may assume that $\|B - B'_m\| \leq \delta$ for $m \gg 0$ by choosing F_m closer to B° if necessary. Therefore, by Lemma 2.8 applied to the polytope $\mathcal{L}(V)$

and the point $B \in \mathcal{L}(V)$, we have $B'_m \in \mathcal{L}(V)$ for $m \gg 0$, and thus $B'_m \in \mathcal{P}_A(V)$ since $K_X+A+B'_m = K_X+A^\circ+F_m$ is numerically equivalent to an effective divisor. This finishes the proof of Claim 5.6.

Step 6. Therefore $\mathcal{P}_A(V)$ is a polytope, and we finally show that it is a *rational* polytope. Let $B_1, \dots, B_q$ be the extreme points of $\mathcal{P}_A(V)$. Then there exist $\mathbb{R}$-divisors $D_i \geq 0$ such that $K_X + A + B_i \equiv D_i$ for all i. Let $W \subseteq \mathrm{Div}_{\mathbb{R}}(X)$ be the vector space spanned by V and by the components of K_X+A and $\sum_{i=1}^q D_i$. Note that for every $\tau = (t_1, \dots, t_q) \in \mathbb{R}^q_+$ such that $\sum t_i = 1$, we have $B_\tau = \sum t_i B_i \in \mathcal{P}_A(V)$ and $K_X+A+B_\tau \equiv \sum t_i D_i \in W$. Let $\phi\colon W \longrightarrow N^1(X)_{\mathbb{R}}$ be the linear map sending an $\mathbb{R}$-divisor to its numerical class. Then $W_0 = \phi^{-1}(0)$ is a rational subspace of W and

$$\mathcal{P}_A(V) = \{B \in \mathcal{L}(V) \mid B = -K_X - A + D + R, \text{ where } 0 \leq D \in W, R \in W_0\}.$$

Therefore, $\mathcal{P}_A(V)$ is cut out from $\mathcal{L}(V) \subseteq W$ by finitely many rational half-spaces, and thus is a rational polytope. □

6. Finite generation

In this section, we prove that Theorem A$_{n-1}$ and Theorem B$_n$ imply Theorem A$_n$; as an immediate consequence, we obtain Theorem 1.1.

Lemma 6.1. *Let $(X, \sum_{i=1}^p S_i)$ be a log smooth projective pair, let $\mathcal{C} \subseteq \sum_{i=1}^p \mathbb{R}_+ S_i \subseteq \mathrm{Div}_{\mathbb{R}}(X)$ be a rational polyhedral cone, and let $\mathcal{C} = \bigcup_{j=1}^p \mathcal{C}_j$ be a rational polyhedral decomposition. Denote $\mathcal{S} = \mathcal{C} \cap \mathrm{Div}(X)$ and $\mathcal{S}_j = \mathcal{C}_j \cap \mathrm{Div}(X)$ for all j. Assume that:*

- *(i) there exists $M > 0$ such that, if $\sum \alpha_i S_i \in \mathcal{C}_j$ for some j and some $\alpha_i \in \mathbb{N}$ with $\sum \alpha_i \geq M$, then $\sum \alpha_i S_i - S_j \in \mathcal{C}$;*
- *(ii) the ring $\mathrm{res}_{S_j} R(X, \mathcal{S}_j)$ is finitely generated for every $j = 1, \dots, p$.*

Then the divisorial ring $R(X, \mathcal{S})$ is finitely generated.

Proof. For every $i = 1, \dots, p$, let $\sigma_i \in H^0(X, \mathcal{O}_X(S_i))$ be a section such that $\mathrm{div}\, \sigma_i = S_i$. Let $\mathfrak{R} \subseteq R(X; S_1, \dots, S_p)$ be the ring spanned by $R(X, \mathcal{S})$ and $\sigma_1, \dots, \sigma_p$, and note that $\mathfrak{R}$ is graded by $\sum_{i=1}^p \mathbb{N} S_i$. By Lemma 2.25(i), it is enough to show that $\mathfrak{R}$ is finitely generated.

For any $\alpha = (\alpha_1, \dots, \alpha_p) \in \mathbb{N}^p$, denote $D_\alpha = \sum \alpha_i S_i$ and $\deg(\alpha) = \sum \alpha_i$, and for a section $\sigma \in H^0(X, \mathcal{O}_X(D_\alpha))$, set $\deg(\sigma) = \deg(\alpha)$. By (ii), for each $j = 1, \dots, p$ there exists a finite set $\mathcal{H}_j \subseteq R(X, \mathcal{S}_j)$ such that

$$\mathrm{res}_{S_j} R(X, \mathcal{S}_j) \quad \text{is generated by the set} \quad \{\sigma_{|S_j} \mid \sigma \in \mathcal{H}_j\}. \tag{66}$$

Since the vector space $H^0(X, \mathcal{O}_X(D_\alpha))$ is finite-dimensional for every $\alpha \in \mathbb{N}^p$, there is a finite set $\mathcal{H} \subseteq \mathfrak{R}$ such that

$$\{\sigma_1, \dots, \sigma_p\} \cup \mathcal{H}_1 \cup \dots \cup \mathcal{H}_p \subseteq \mathcal{H}, \tag{67}$$

and

$$H^0(X, \mathcal{O}_X(D_\alpha)) \subseteq \mathbb{C}[\mathcal{H}] \quad \text{for every } \alpha \in \mathbb{N}^p \text{ with } D_\alpha \in \mathcal{S} \text{ and } \deg(\alpha) \leq M, \tag{68}$$

where $\mathbb{C}[\mathcal{H}]$ is the $\mathbb{C}$-algebra generated by the elements of $\mathcal{H}$. Observe that $\mathbb{C}[\mathcal{H}] \subseteq \mathfrak{R}$, and it suffices to show that $\mathfrak{R} \subseteq \mathbb{C}[\mathcal{H}]$.

Let $\chi \in \mathfrak{R}$. By definition of $\mathfrak{R}$, we may write $\chi = \sum_i \sigma_1^{\lambda_{1,i}} \ldots \sigma_p^{\lambda_{p,i}} \chi_i$, where $\chi_i \in H^0(X, \mathcal{O}_X(D_{\alpha_i}))$ for some $D_{\alpha_i} \in \mathcal{S}$ and $\lambda_{j,i} \in \mathbb{N}$. Thus, it is enough to show that $\chi_i \in \mathbb{C}[\mathcal{H}]$, and after replacing χ by χ_i we may assume that

$$\chi \in H^0(X, \mathcal{O}_X(D_\alpha)) \quad \text{for some} \quad D_\alpha \in \mathcal{S}.$$

The proof is by induction on $\deg \chi$. If $\deg \chi \leq M$, then $\chi \in \mathbb{C}[\mathcal{H}]$ by (68). Now assume $\deg \chi > M$. Then there exists $1 \leq j \leq p$ such that $D_\alpha \in \mathcal{S}_j$, and so by (66) and (67) there are $\theta_1, \ldots, \theta_z \in \mathcal{H}$ and a polynomial $\varphi \in \mathbb{C}[X_1, \ldots, X_z]$ such that $\chi_{|S_j} = \varphi(\theta_{1|S_j}, \ldots, \theta_{z|S_j})$. Therefore, from the exact sequence

$$0 \longrightarrow H^0(X, \mathcal{O}_X(D_\alpha - S_j)) \xrightarrow{\cdot \sigma_j} H^0(X, \mathcal{O}_X(D_\alpha)) \longrightarrow H^0(S_j, \mathcal{O}_{S_j}(D_\alpha))$$

we obtain

$$\chi - \varphi(\theta_1, \ldots, \theta_z) = \sigma_j \cdot \chi' \quad \text{for some} \quad \chi' \in H^0(X, \mathcal{O}_X(D_\alpha - S_j)).$$

Note that $D_\alpha - S_j \in \mathcal{S}$ by (i), and since $\deg \chi' < \deg \chi$, by induction we have $\chi' \in \mathbb{C}[\mathcal{H}]$. Therefore $\chi = \sigma_j \cdot \chi' + \varphi(\theta_1, \ldots, \theta_z) \in \mathbb{C}[\mathcal{H}]$, and we are done. $\square$

Lemma 6.2. *Assume Theorem A_{n-1} and Theorem B_{n-1}.*

Let $(X, S + \sum_{i=1}^p S_i)$ be a log smooth projective pair of dimension n, where S and all S_i are distinct prime divisors. Let $V = \sum_{i=1}^p \mathbb{R} S_i \subseteq \operatorname{Div}_{\mathbb{R}}(X)$, let A be an ample $\mathbb{Q}$-divisor on X, let $B_1, \ldots, B_m \in \mathcal{E}_{S+A}(V)$ be $\mathbb{Q}$-divisors, and denote $D_i = K_X + S + A + B_i$.

Then the ring $\operatorname{res}_S R(X; D_1, \ldots, D_m)$ is finitely generated.

Proof. We first prove the lemma under an additional assumption that all B_i lie in the interior of $\mathcal{L}(V)$ and that all $(S, B_{i|S})$ are terminal, and then treat the general case at the end of the proof.

Let $\mathcal{G} \subseteq \mathcal{E}_{S+A}(V)$ be the convex hull of all B_i. Then $\mathcal{G}$ is contained in the interior of $\mathcal{L}(V)$, and $(S, G_{|S})$ is terminal for every $G \in \mathcal{G}$. Denote

$$\mathcal{F} = \mathbb{R}_+(K_X + S + A + \mathcal{G}).$$

Then, by Lemma 2.27 it suffices to prove that $\operatorname{res}_S R(X, \mathcal{F})$ is finitely generated.

Let $W \subseteq \operatorname{Div}_{\mathbb{R}}(S)$ be the subspace spanned by the components of all $S_{i|S}$, and let Φ_m and $\mathbf{\Phi}$ be the functions defined in Setup 4.1. By Theorem 4.3, $\mathcal{P} = \mathcal{G} \cap \mathcal{B}_A^S(V)$ is a rational polytope, and $\mathbf{\Phi}$ extends to a rational piecewise affine function on $\mathcal{P}$. Thus, there exists a finite decomposition $\mathcal{P} = \bigcup \mathcal{P}_i$ into rational polytopes such that $\mathbf{\Phi}$ is rational affine on each $\mathcal{P}_i$. Denote $\mathcal{C} = \mathbb{R}_+(K_X + S + A + \mathcal{P})$ and $\mathcal{C}_i = \mathbb{R}_+(K_X + S + A + \mathcal{P}_i)$, and note that $\mathcal{C} = \bigcup \mathcal{C}_i$. Since $\operatorname{res}_S H^0(X, \mathcal{O}_X(D)) = 0$ for every $D \in \mathcal{F} \setminus \mathcal{C}$, and as $\mathcal{C}$ is a rational polyhedral cone, it suffices to show that $\operatorname{res}_S R(X,\mathcal{C})$ is finitely generated, and therefore, to prove that $\operatorname{res}_S R(X,\mathcal{C}_i)$ is finitely generated for each i. Hence, after replacing $\mathcal{G}$ by $\mathcal{P}_i$, we can assume that $\mathbf{\Phi}$ is rational affine on $\mathcal{G}$.

By Lemma 2.11, there exist $G_i \in \mathcal{G} \cap \operatorname{Div}_{\mathbb{Q}}(X)$ and $g_i \in \mathbb{Q}_+$, with $i = 1, \dots, q$, such that $F_i = g_i(K_X + S + A + G_i)$ are generators of $\mathcal{F} \cap \operatorname{Div}(X)$. By Theorem 4.3, there exists a positive integer ℓ such that $\Phi_m(G) = \mathbf{\Phi}(G)$ for every $G \in \mathcal{G} \cap \operatorname{Div}_{\mathbb{Q}}(X)$ and every $m \in \mathbb{N}$ such that $\frac{m}{\ell} G \in \operatorname{Div}(X)$. Pick a positive integer k such that all $\frac{kg_i}{\ell} \in \mathbb{N}$ and $\frac{kg_i}{\ell} G_i \in \operatorname{Div}(X)$. For each nonzero $\alpha = (\alpha_1, \dots, \alpha_q) \in \mathbb{N}^q$, denote

$$g_\alpha = \sum \alpha_i g_i, \quad G_\alpha = \frac{1}{g_\alpha} \sum \alpha_i g_i G_i, \quad F_\alpha = \sum \alpha_i F_i = g_\alpha(K_X + S + A + G_\alpha),$$

and note that $\frac{kg_\alpha}{\ell} G_\alpha \in \operatorname{Div}(X)$ and $\mathbf{\Phi}(G_\alpha) = \frac{1}{g_\alpha} \sum \alpha_i g_i \mathbf{\Phi}(G_i)$. Then, by Corollary 3.5 we have

$$\begin{aligned} \operatorname{res}_S H^0(X, \mathcal{O}_X(mkF_\alpha)) &= H^0\big(S, \mathcal{O}_S(mkg_\alpha(K_S + A_{|S} + \Phi_{mkg_\alpha}(G_\alpha)))\big) \\ &= H^0\big(S, \mathcal{O}_S(mkg_\alpha(K_S + A_{|S} + \mathbf{\Phi}(G_\alpha)))\big) \end{aligned}$$

for all $\alpha \in \mathbb{N}^q$ and $m \in \mathbb{N}$, and thus

$$\operatorname{res}_S R(X; kF_1, \dots, kF_q) = R(S; kg_1 F_1', \dots, kg_q F_q'),$$

where $F_i' = K_S + A_{|S} + \mathbf{\Phi}(G_i)$. Since the last ring is a Veronese subring of the adjoint ring $R(S; F_1', \dots, F_q')$, it is finitely generated by Theorem A_{n-1} and by Lemma 2.25(i). Therefore $\operatorname{res}_S R(X; F_1, \dots, F_q)$ is finitely generated by Lemma 2.25(ii), and since there is the natural projection of this ring onto $\operatorname{res}_S R(X, \mathcal{F})$, this proves the lemma under the additional assumption that all B_i lie in the interior of $\mathcal{L}(V)$ and that all $(S, B_{i|S})$ are terminal.

In the general case, for every i pick a $\mathbb{Q}$-divisor $G_i \in V$ such that $A - G_i$ is ample and $B_i + G_i$ is in the interior of $\mathcal{L}(V)$. Let A' be an ample $\mathbb{Q}$-divisor such that every $A - G_i - A'$ is also ample, and pick $\mathbb{Q}$-divisors $A_i \geq 0$ such that $A_i \sim_{\mathbb{Q}} A - G_i - A'$, $\lfloor A_i \rfloor = 0$, $(X, S + \sum_{i=1}^p S_i + \sum_{i=1}^m A_i)$ is log smooth, and the support of $\sum_{i=1}^m A_i$ does not contain any of the divisors $S, S_1, \dots, S_p$. Let $V' \subseteq \operatorname{Div}_{\mathbb{R}}(X)$ be the vector space spanned by V and by the components of $\sum_{i=1}^m A_i$. Let $\varepsilon > 0$ be a rational number such that $A'' = A' - \varepsilon \sum_{i=1}^m A_i$ is ample, and such that $B_i' = B_i + G_i + A_i + \varepsilon \sum_{i=1}^m A_i$ is in the interior of $\mathcal{L}(V')$ for every i.

Let $B \geq 0$ be a $\mathbb{Q}$-divisor such that $\lfloor B \rfloor = 0$ and $B \geq B_i'$ for all i. By Lemma 2.2, there exists a log resolution $f\colon Y \longrightarrow X$ such that

$$K_Y + T + C = f^*(K_X + S + B) + E,$$

where the $\mathbb{Q}$-divisors $C, E \geq 0$ have no common components, E is f-exceptional, $\lfloor C \rfloor = 0$, the components of C are disjoint, and $T = f_*^{-1} S \nsubseteq \operatorname{Supp} C$. Then there are $\mathbb{Q}$-divisors $0 \leq C_i \leq C$ and f-exceptional $\mathbb{Q}$-divisors $E_i \geq 0$ such that

$$K_Y + T + C_i = f^*(K_X + S + B_i') + E_i,$$

and in particular, all pairs $(T, C_{i|T})$ are terminal. Let V° be the subspace of $\mathrm{Div}_{\mathbb{R}}(Y)$ spanned by the components of C and by all f-exceptional prime divisors. There exists a $\mathbb{Q}$-divisor $F \geq 0$ on Y such that A° is ample, every C_i° is in the interior of $\mathcal{L}(V^\circ)$, and every pair $(T, C^\circ_{i|T})$ is terminal, where $A^\circ = f^*A'' - F$ and $C_i^\circ = C_i + F$. Denoting $D_i^\circ = K_Y + T + A^\circ + C_i^\circ$, it follows that

$$D_i^\circ \sim_{\mathbb{Q}} f^*D_i + E_i.$$

Then the ring $\mathrm{res}_T\, R(Y; D_1^\circ, \dots, D_m^\circ)$ is finitely generated by the special case that we proved above, so $\mathrm{res}_S\, R(X; D_1, \dots, D_m)$ is finitely generated by Corollary 2.26. $\square$

Theorem 6.3. *Theorem A_{n-1} and Theorem B_n imply Theorem A_n.*

Proof. We first assume that there exist $\mathbb{Q}$-divisors $F_i \geq 0$ such that

(69) $\quad \big(X, \sum_i (B_i + F_i)\big)$ is log smooth and $K_X + A + B_i \sim_{\mathbb{Q}} F_i$ for every i.

We reduce the general case to this one at the end of the proof.

Let W be the subspace of $\mathrm{Div}_{\mathbb{R}}(X)$ spanned by the components of all B_i and F_i, and let $S_1, \dots, S_p$ be the prime divisors in W. Denote by $\mathcal{T} = \{(t_1, \dots, t_k) \mid t_i \geq 0, \sum t_i = 1\} \subseteq \mathbb{R}^k$ the standard simplex, and for each $\tau = (t_1, \dots, t_k) \in \mathcal{T}$, set

$$B_\tau = \sum_{i=1}^k t_i B_i \qquad \text{and} \qquad F_\tau = \sum_{i=1}^k t_i F_i \sim_{\mathbb{R}} K_X + A + B_\tau. \tag{70}$$

Denote

$$\mathcal{B} = \{F_\tau + B \mid \tau \in \mathcal{T}, 0 \leq B \in W, B_\tau + B \in \mathcal{L}(W)\} \subseteq W,$$

and for every $j = 1, \dots, p$, let

$$\mathcal{B}_j = \{F_\tau + B \mid \tau \in \mathcal{T}, 0 \leq B \in W, B_\tau + B \in \mathcal{L}(W), S_j \subseteq \lfloor B_\tau + B \rfloor\} \subseteq W.$$

Then $\mathcal{B}$ and $\mathcal{B}_j$ are rational polytopes, and thus $\mathcal{C} = \mathbb{R}_+\mathcal{B}$ and $\mathcal{C}_j = \mathbb{R}_+\mathcal{B}_j$ are rational polyhedral cones. Denote $\mathcal{S} = \mathcal{C} \cap \mathrm{Div}(X)$ and $\mathcal{S}_j = \mathcal{C}_j \cap \mathrm{Div}(X)$. We claim that:

(i) $\mathcal{C} = \bigcup_{j=1}^p \mathcal{C}_j$,
(ii) there exists $M > 0$ such that, if $\sum \alpha_i S_i \in \mathcal{C}_j$ for some j and some $\alpha_i \in \mathbb{N}$ with $\sum \alpha_i \geq M$, then $\sum \alpha_i S_i - S_j \in \mathcal{C}$;
(iii) the ring $\mathrm{res}_{S_j}\, R(X, \mathcal{S}_j)$ is finitely generated for every $j = 1, \dots, p$.

This claim readily implies the theorem: indeed, Lemma 6.1 then shows that $R(X, \mathcal{S})$ is finitely generated. Let d be a positive integer such that $F_i' = dF_i$ are integral divisors for $i = 1, \dots, k$. Pick divisors $F'_{k+1}, \dots, F'_m$ such that $F'_1, \dots, F'_m$ are generators of $\mathcal{S}$. Then $R(X; F'_1, \dots, F'_m)$ is finitely generated by Lemma 2.27, and so is $R(X; F'_1, \dots, F'_k)$ by Lemma 2.25(i). Finally, Lemma 2.25(ii) implies that $R(X; F_1, \dots, F_k)$ is finitely generated, and therefore so is $R(X; D_1, \dots, D_k)$ by (69) and by Corollary 2.26.

We now prove the claim. In order to see (i), fix $G \in \mathcal{C}\backslash\{0\}$. Then, by definition of $\mathcal{C}$, there exist $\tau \in \mathcal{T}, 0 \leq B \in W$ and $r > 0$ such that $B_\tau + B \in \mathcal{L}(W)$ and $G = r(F_\tau + B)$. Setting

$$\lambda = \max\{t \geq 1 \mid B_\tau + tB + (t-1)F_\tau \in \mathcal{L}(W)\}$$

and $B' = \lambda B + (\lambda - 1)F_\tau$, we have

$$\lambda G = r(F_\tau + B'),$$

and there exists j_0 such that $S_{j_0} \subseteq \lfloor B_\tau + B' \rfloor$. Therefore $G \in \mathcal{C}_{j_0}$, which proves (i).

For (ii), note first that there exists $\varepsilon > 0$ such that $\|B_i\| \leq 1 - \varepsilon$ for all i, and thus

$$\|B_\tau\| \leq 1 - \varepsilon \quad \text{for any } \tau \in \mathcal{T}. \tag{71}$$

Since the polytopes $\mathcal{B}_j \subseteq W$ are compact, there is a positive constant C such that $\|\Psi\| \leq C$ for any $\Psi \in \bigcup_{j=1}^p \mathcal{B}_j$, and denote $M = pC/\varepsilon$. For some $j \in \{1, \ldots, p\}$, let $G = \sum \alpha_i S_i \in \mathcal{S}_j$ be such that $\sum \alpha_i \geq M$. Since $p\|G\| \geq \sum \alpha_i$, we have

$$\|G\| \geq \frac{M}{p} = \frac{C}{\varepsilon}.$$

By definition of $\mathcal{C}_j$ and of C, we may write $G = rG'$ with $G' \in \mathcal{B}_j$, $\|G'\| \leq C$ and $r > 0$. In particular,

$$r = \frac{\|G\|}{\|G'\|} \geq \frac{1}{\varepsilon}. \tag{72}$$

Furthermore, $G' = F_\tau + B$ for some $\tau \in \mathcal{T}$ and $0 \leq B \in W$ such that $B_\tau + B \in \mathcal{L}(W)$ and $S_j \subseteq \lfloor B_\tau + B \rfloor$. Therefore, by (71) and (72) we have

$$\operatorname{mult}_{S_j} B = 1 - \operatorname{mult}_{S_j} B_\tau \geq \varepsilon \geq \frac{1}{r},$$

and thus

$$G - S_j = r\big(F_\tau + B - \tfrac{1}{r} S_j\big) \in \mathcal{C}.$$

Finally, to show (iii), fix $j \in \{1, \ldots, p\}$, and let $\{E_1, \ldots, E_\ell\}$ be a set of generators of $\mathcal{S}_j$. Then, by definition of $\mathcal{S}_j$ and by (70), for every $i = 1, \ldots, \ell$, there exist $k_i \in \mathbb{Q}_+$, $\tau_i \in \mathcal{T} \cap \mathbb{Q}^k$ and $0 \leq B_i \in W$ such that $B_{\tau_i} + B_i \in \mathcal{L}(W)$, $S_j \subseteq \lfloor B_{\tau_i} + B_i \rfloor$ and

$$E_i = k_i(F_{\tau_i} + B_i) \sim_{\mathbb{Q}} k_i(K_X + A + B_{\tau_i} + B_i).$$

Denote $E_i' = K_X + A + B_{\tau_i} + B_i$. Then the ring $\operatorname{res}_{S_j} R(X; E_1', \ldots, E_\ell')$ is finitely generated by Lemma 6.2, and thus so is $\operatorname{res}_{S_j} R(X; E_1, \ldots, E_\ell)$ by Corollary 2.26. Since there is the natural projection $\operatorname{res}_{S_j} R(X; E_1, \ldots, E_\ell) \longrightarrow \operatorname{res}_{S_j} R(X, \mathcal{S}_j)$, this completes the proof under the additional assumption that (69) holds.

We now prove the general case. Let V be the subspace of $\operatorname{Div}_{\mathbb{R}}(X)$ spanned by the components of all B_i, let $\mathcal{P} \subseteq V$ be the convex hull of all B_i, and denote $\mathcal{R} = \mathbb{R}_+(K_X + A + \mathcal{P})$. Then, by Lemma 2.27 it suffices to show that $R(X, \mathcal{R})$ is finitely generated. By Theorem B_n, $\mathcal{P}_\mathcal{E} = \mathcal{P} \cap \mathcal{E}_A(V)$ is a rational polytope, and denote $\mathcal{R}_\mathcal{E} = \mathbb{R}_+(K_X + A + \mathcal{P}_\mathcal{E})$. Since $H^0(X, \mathcal{O}_X(D)) = 0$ for every integral divisor $D \in \mathcal{R} \setminus \mathcal{R}_\mathcal{E}$, the ring $R(X, \mathcal{R})$ is finitely generated if and only if $R(X, \mathcal{R}_\mathcal{E})$ is.

By Lemma 2.11, the monoid $\mathcal{R}_\mathcal{E} \cap \mathrm{Div}(X)$ is finitely generated, and let R_i be its generators for $i = 1, \dots, \ell$. Then there exist $p_i \in \mathbb{Q}_+$ and $P_i \in \mathcal{P}_\mathcal{E} \cap \mathrm{Div}_\mathbb{Q}(X)$ such that $R_i = p_i(K_X + A + P_i)$. By construction, $\lfloor P_i \rfloor = 0$ and there exist $\mathbb{Q}$-divisors $G_i \geq 0$ such that

$$K_X + A + P_i \sim_\mathbb{Q} G_i$$

for all i. Let $f\colon Y \longrightarrow X$ be a log resolution of $\big(X, \sum_i (P_i + G_i)\big)$. For every i, there are $\mathbb{Q}$-divisors $C_i, E_i \geq 0$ on Y with no common components such that E_i is f-exceptional and

$$K_Y + C_i = f^*(K_X + P_i) + E_i.$$

Note that $\lfloor C_i \rfloor = 0$, and denote $F_i^\circ = p_i(f^*G_i + E_i) \geq 0$. Let $H \geq 0$ be an f-exceptional $\mathbb{Q}$-divisor on Y such that A° is ample and $\lfloor C_i^\circ \rfloor = 0$ for all i, where $A^\circ = f^*A - H$ is ample and $C_i^\circ = C_i + H$, and denote $D_i^\circ = K_Y + A^\circ + C_i^\circ$. Then

$$p_i D_i^\circ \sim_\mathbb{Q} f^*R_i + p_i E_i \sim_\mathbb{Q} F_i^\circ.$$

This last relation implies two things: first, it follows from what we proved above and by Lemma 2.25 that the adjoint ring $R(Y; D_1^\circ, \dots, D_\ell^\circ)$ is finitely generated. Second, the ring $R(X; R_1, \dots, R_\ell)$ is then finitely generated by Corollary 2.26. Since there is the natural projection map $R(X; R_1, \dots, R_\ell) \longrightarrow R(X, \mathcal{R}_\mathcal{E})$, the ring $R(X, \mathcal{R}_\mathcal{E})$ is finitely generated, and we are done. $\square$

Finally, we have:

Proof of Theorem 1.1. By Theorem 2.29, there exist a projective klt pair (Y, Γ) and positive integers p and q such that $p(K_X + \Delta)$ and $q(K_Y + \Gamma)$ are integral, $K_Y + \Gamma$ is big and $R(X, p(K_X + \Delta)) \simeq R(Y, q(K_Y + \Gamma))$. Write $K_Y + \Gamma \sim_\mathbb{Q} A + B$, where A is an ample $\mathbb{Q}$-divisor and $B \geq 0$. Let $f\colon Y' \longrightarrow Y$ be a log resolution of $(Y, \Gamma + B)$, let $\Gamma', E \geq 0$ be $\mathbb{Q}$-divisors such that E is f-exceptional and $K_{Y'} + \Gamma' = f^*(K_Y + \Gamma) + E$, and let $H \geq 0$ be an f-exceptional $\mathbb{Q}$-divisor such that $A' = f^*A - H$ is ample. Pick a rational number $0 < \varepsilon \ll 1$ such that if $C = \Gamma' + \varepsilon f^*B + \varepsilon H$, then $\lfloor C \rfloor = 0$, and note that $K_{Y'} + C + \varepsilon A' \sim_\mathbb{Q} (\varepsilon + 1) f^*(K_Y + \Gamma) + E$. Then the ring $R(Y, K_Y + \Gamma)$ is finitely generated by Theorem A and Corollary 2.26, and thus so is $R(X, K_X + \Delta)$ by Lemma 2.25. $\square$

References

[ADHL10] I. Arzhantsev, U. Derenthal, J. Hausen, and A. Laface, *Cox Rings*, arXiv:1003.4229v1.

[BCHM10] C. Birkar, P. Cascini, C. Hacon, and J. McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. **23** (2010), no. 2, 405–468.

[Bou89] N. Bourbaki, *Commutative algebra. Chapters 1–7*, Elements of Mathematics, Springer-Verlag, Berlin, 1989.

[CL10] A. Corti and V. Lazić, *New outlook on the Minimal Model Program, II*, arXiv:1005.0614v2.

[CL11] P. Cascini and V. Lazić, *The Minimal Model Program Revisited*, to appear in Contributions to Algebraic Geometry (P. Pragacz, ed.), EMS Publishing House.

[Cor07] A. Corti, *3-fold flips after Shokurov*, Flips for 3-folds and 4-folds (A. Corti, ed.), Oxford University Press, 2007, pp. 13–40.

[Cor11] ———, *Finite generation of adjoint rings after Lazić: an introduction*, Classification of Algebraic Varieties (C. Faber, G. van der Geer, and E. Looijenga, eds.), EMS Series of Congress Reports, EMS Publishing House, 2011, pp. 197–220.

[ELM+06] L. Ein, R. Lazarsfeld, M. Mustaţă, M. Nakamaye, and M. Popa, *Asymptotic invariants of base loci*, Ann. Inst. Fourier (Grenoble) **56** (2006), no. 6, 1701–1734.

[FM00] O. Fujino and S. Mori, *A canonical bundle formula*, J. Differential Geom. **56** (2000), no. 1, 167–188.

[Ful93] W. Fulton, *Introduction to toric varieties*, Princeton University Press, 1993.

[HK10] C. Hacon and S. Kovács, *Classification of Higher Dimensional Algebraic Varieties*, Oberwolfach Seminars, vol. 41, Birkhäuser, Basel, 2010.

[HM10] C. Hacon and J. McKernan, *Existence of minimal models for varieties of log general type II*, J. Amer. Math. Soc. **23** (2010), no. 2, 469–490.

[KM98] J. Kollár and S. Mori, *Birational Geometry of Algebraic Varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, 1998.

[Laz09] V. Lazić, *Adjoint rings are finitely generated*, arXiv:0905.2707v3.

[Mor82] S. Mori, *Threefolds whose canonical bundles are not numerically effective*, Ann. of Math. (2) **116** (1982), no. 1, 133–176.

[Nak04] N. Nakayama, *Zariski-decomposition and abundance*, MSJ Memoirs, vol. 14, Mathematical Society of Japan, Tokyo, 2004.

[Pău08] M. Păun, *Relative critical exponents, non-vanishing and metrics with minimal singularities*, arXiv:0807.3109v1.

[Sho03] V. V. Shokurov, *Prelimiting flips*, Proc. Steklov Inst. of Math. **240** (2003), 82–219.

[Siu98] Y.-T. Siu, *Invariance of plurigenera*, Invent. Math. **134** (1998), no. 3, 661–673.

[Siu08] ———, *Finite generation of canonical ring by analytic method*, Sci. China Ser. A **51** (2008), no. 4, 481–502.

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